ME 406
The van der Pol Oscillator

■ 1. Introduction

The labels mathematician, engineer, and physicist have all been used in reference to Balthasar van der Pol. The van der Pol oscillator, which we study in this notebook, is a model developed by him to describe the behavior of nonlinear vacuum tube circuits in the relatively early days of the development of electronics technology. A little more detail on his work, taken from the Exploratorium web site (http://www.exploratorium.edu), is given below. A brief description of a circuit described by the van der Pol equation is given in **Nonlinear Dynamics and Chaos**, Steven Strogatz, Addison-Wesley 1994, p. 228. Chapter 7 of Strogatz' book contains a very readable discussion of the equation. Our study in this notebook will be based entirely on numerical solutions. The rigorous foundations for the analysis (e.g., the proof that the equation has a limit cycle solution which is a global attractor) date back to the work of Lienard in 1928, with later more general analysis by Levinson and others. Perturbation techniques are also useful for the case of large parameter, but we will not consider them here.

From Exploratorium web site:

"Balthazar van der Pol was a Dutch electrical engineer who initiated modern experimental dynamics in the laboratory during the 1920's and 1930's. Van der Pol investigated electrical circuits employing vacuum tubes and found that they have stable oscillations, now called limit cycles. When these circuits are driven with a signal whose frequency is near that of the limit cycle, the resulting periodic response shifts its frequency to that of the driving signal.That is to say,the circuit becomes "entrained" to the driving signal. The waveform, or signal shape, however, can be quite complicated and contain a rich structure of harmonics and subharmonics. In the September 1927 issue of the British journal Nature, he and his colleague van der Mark reported that an "irregular noise" was heard at certain driving frequencies between the natural entrainment frequencies. By reconstructing his electronic tube circuit, we now know that they had discovered deterministic chaos. Their paper is probably one of the first experimental reports of chaos --- something that they failed to pursue in more detail. Van der Pol built a number of electronic circuit models of the human heart to study the range of stability of heart dynamics. His investigations with adding an external driving signal were analogous to the situation in which a real heart is driven by a pacemaker. He was interested in finding out, using his entrainment work, how to stabilize a heart's irregular beating or "arrhythmias"."

Balthazar van der Pol (1889-1959)

Picture from Modern Differential Equations, Martha L. Abell andJames P. Braselton, Saunders, 1996.

■ 2. The van der Pol Equation

The van der Pol equation, in what is now considered to be standard form, is given by

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \; . \qquad (1)$$

We see that it is an oscillator with a linear spring force and a nonlinear damping force. In all that follows, we take $\mu > 0$. The time in the equation has been scaled so that the frequency associated with the spring force alone is unity. The damping force varies in an interesting way. For $|x| < 1$, the damping is actually negative and hence produces an amplification of the motion. For $|x| > 1$, there is true damping and the motion decays. These observations suggest the possibility of an oscillation, in which the system starts at small *x,* is driven to large *x* by the amplification, and is then damped back to small *x*. We will explore this possibility by using DynPac to construct orbits. We define the equation for DynPac, after converting it to the following system:

$$\dot{x} = y, \;\; \dot{y} = -x - \mu(x^2 - 1)y \; . \qquad (2)$$

```
sysid
```

```
Mathematica  5.2.0, DynPac 10.71, 10/3/2005
```

```
intreset;
```

```
plotreset;
```

```
setstate[{x, y}]; setparm[{μ}]; slopevec = {y, -x - μ (x^2 - 1) y};
```

```
sysname = "van der Pol";
```

For our initial explorations, we take the parameter $\mu = 1$.

```
parmval = {1};
```

■ 3. Equilibrium and Stability

There is an equilibrium at the origin, and it is obvious from the slope vector that there are no other equilibria. We give this equilibrium a name, and then look at the eigenvalues of the derivative matrix eqmat at the equilibrium.

```
eq = {0, 0};
```

```
eqmat = dermat /. Thread[statevec → eq]
```

```
{{0, 1}, {-1, μ}}
```

```
Eigenvalues[eqmat]
```

$$\left\{\frac{1}{2}\left(\mu-\sqrt{-4+\mu^2}\right), \frac{1}{2}\left(\mu+\sqrt{-4+\mu^2}\right)\right\}$$

We see that if $0 < \mu < 2$, the equilibrium is an unstable spiral. For $\mu > 2$, the equilibrium is an unstable node. Because there are no other equilibria, the only possible attractors for this system are the point at infinity, or periodic solutions. We will let the computer tell us which.

■ 4. The Limit Cycle

We begin our numerical work with a phase portrait based on four selected initial conditions.

```
initset = {{0.5, 0}, {-0.5, 0}, {3.0, 0}, {-3.0, 0}};
```

We set the integration parameters.

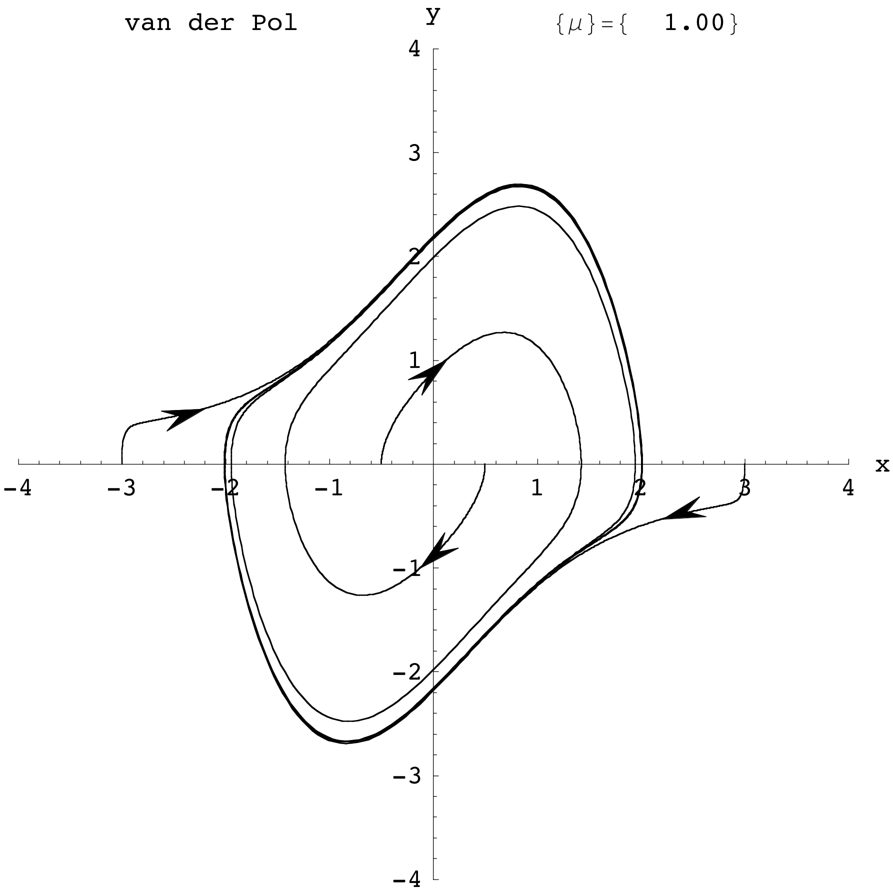
```
t0 = 0.0; h = 0.02; nsteps = 500;
```

We set the plotting parameters.

```
asprat = 1.0; plrange = {{-4, 4}, {-4, 4}}; labshift = 15;
```

```
arrowflag = True; arrowvec = {1 / 15};
```

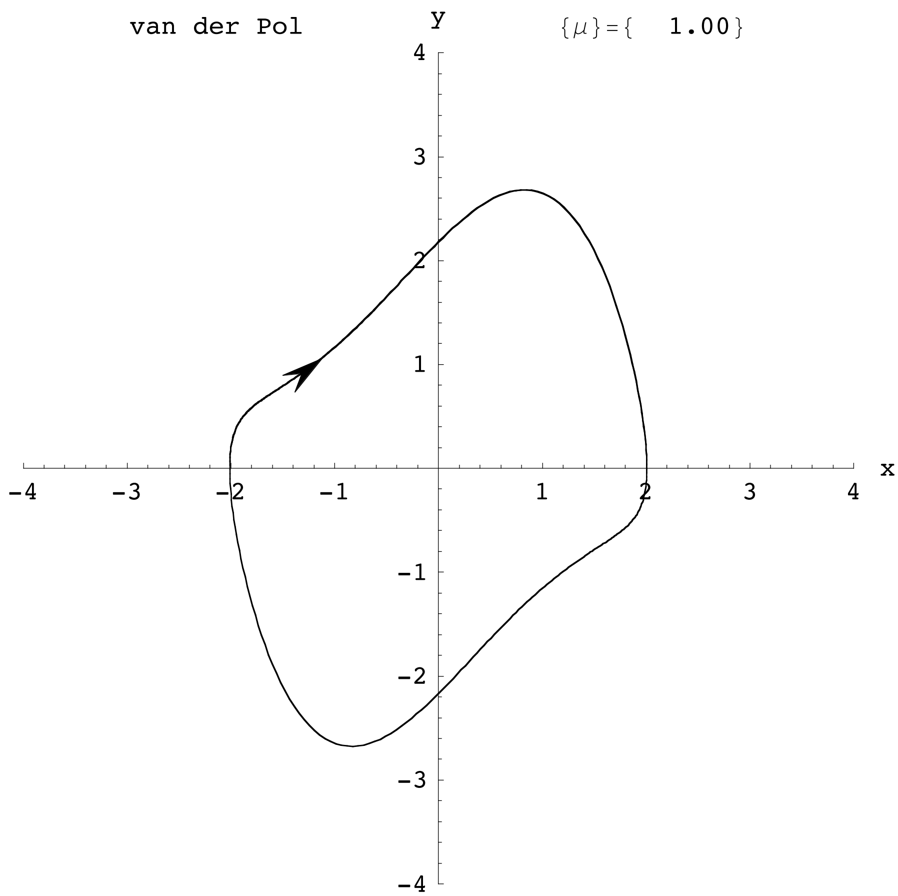
```
graph1 = portrait[initset, t0, h, nsteps, 1, 2];
```



This looks very much like a globally attracting limit cycle. Let's try to construct the pure limit cycle.

```
sol2 = limcyc[{2, 0}, t0, h, nsteps];
```

```
graph2 = phaser[sol2];
```




```
period[sol2]
```

```
6.66
```

The remarkable fact about this oscillator is that every initial condition in the phase plane ultimately leads to this periodic motion with a period of 6.66. (Our calculations only suggest the truth of that statement. A proof requires a rigorous mathematical analysis which is given in some of the references mentioned above.)

■ 5. Dependence of the Limit Cycle on the Parameter

How does the shape of this limit cycle change as the parameter μ changes? We construct a sequence of limit cycles with the parameter μ varying from 0 to 3. We begin by looking at the two end-points of that μ-range to get an idea of the period range and the range of function values in the graph. We first define a function cycgraph[μval] which returns a graph of the cycle for $\mu = \mu$val.

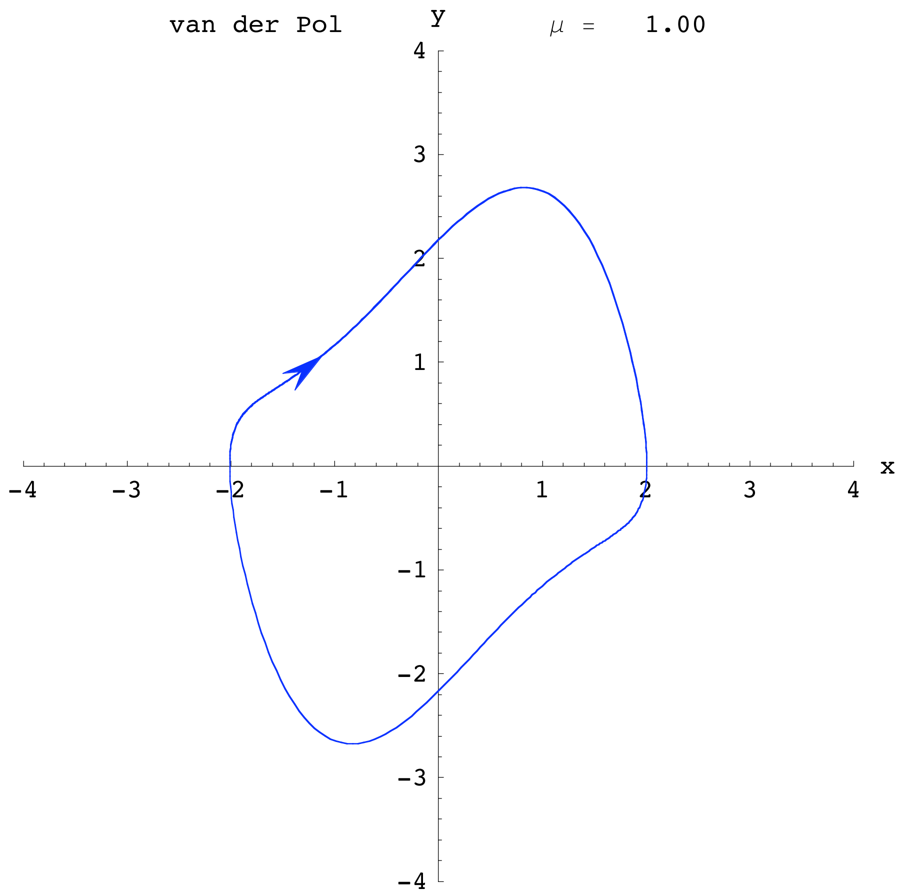
```
setcolor[{Blue}];
```

```
parmval = {0};
```

```
cycgraph[μval_] := (parmval = {μval};
  labon = SequenceForm["van der Pol                μ = ",
    PaddedForm[First[parmval], {4, 2}]];
  phaser[limcyc[{2, 0}, t0, h, nsteps]])
```

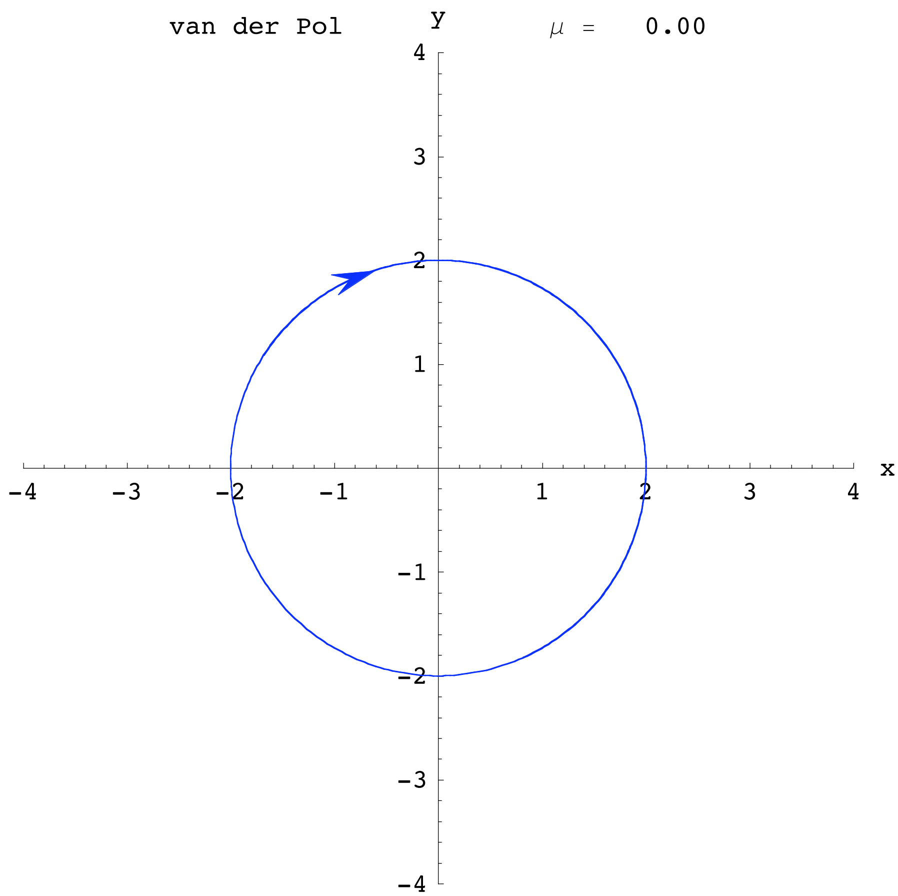
We try out our function on the graph we just constructed.

```
cycgraph[1];
```



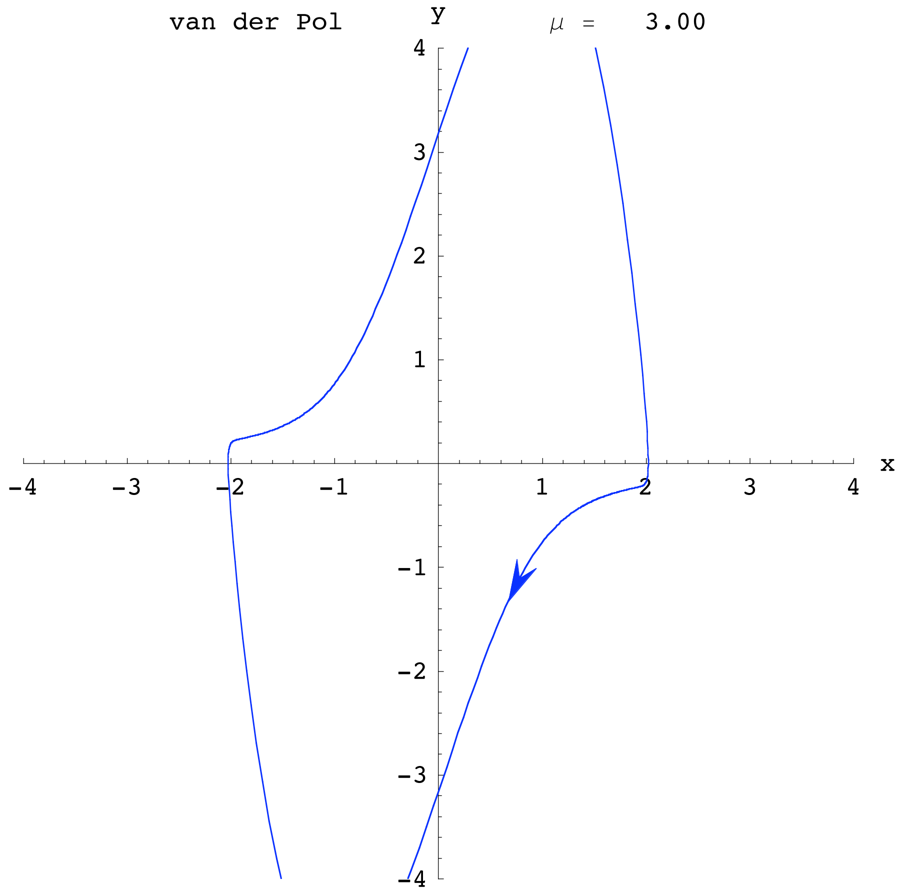
Now the smallest and largest μ-values in our range.

```
cycgraph[0];
```



For $\mu = 0$ we get a circle, as we easily could have predicted by looking at equation (1) for $\mu = 0$. Now $\mu = 3$.

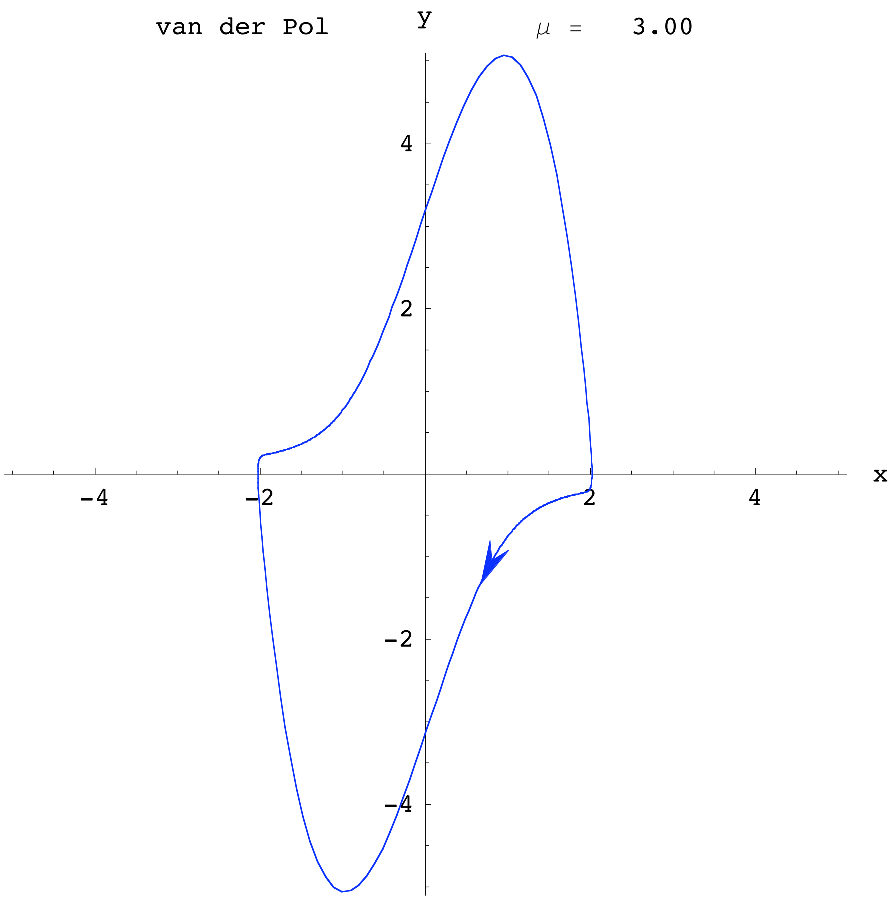
```
cycgraph[3];
```



We see that the range of y for $\mu = 3$ exceeds our graph range. We set a larger plotting window and try again.

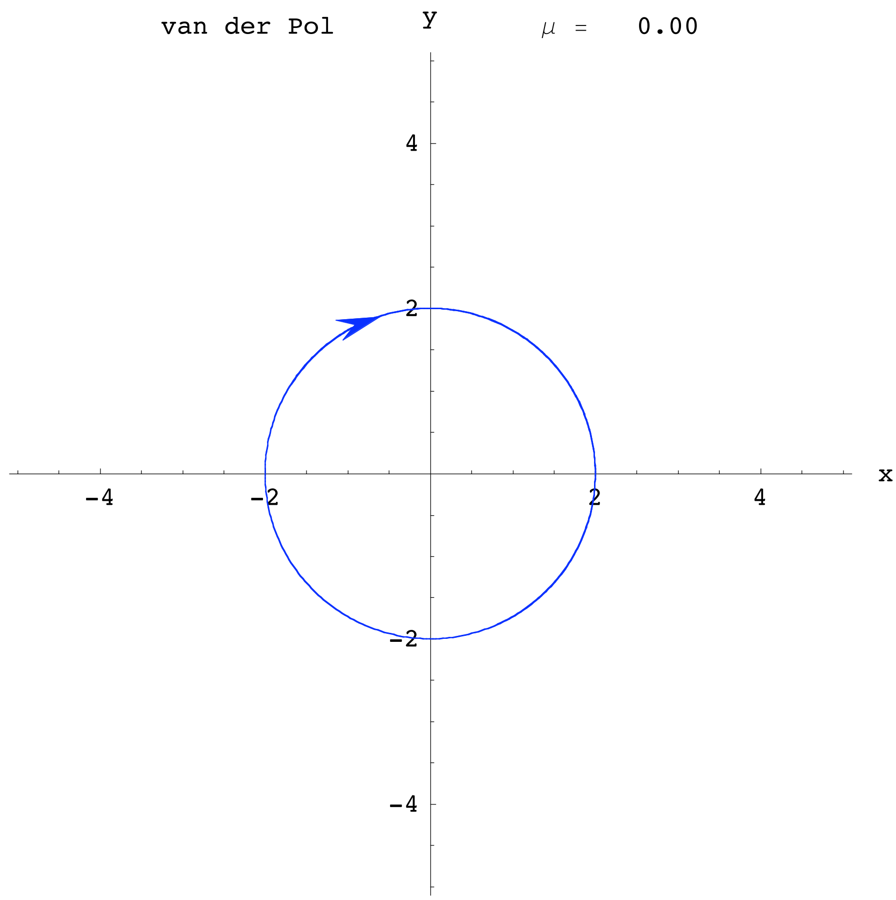
```
plrange = {{-5.1, 5.1}, {-5.1, 5.1}};
```

```
cycgraph[3];
```



Now we create a sequence of graphs suitable for animation. In the printed version of the notebook, only the first graph of the sequence is shown. To start the movie, double click on the graph below.

```
Do[(μval = 0.02 * i; cycgraph[μval]), {i, 0, 150}]
```



■ 6. Real-Time Orbits

The shapes of the orbits for the larger μ values are both surprising and interesting. We can get some insight into those shapes by constructing graph sequences which show a real-time traversal of the orbit. We do that using the function phaseseq.

As we know, the van der Pol oscillator is amplified for $|x| < 1$ and damped for $|x| > 1$. It is helpful to show those lines on the graph sequence, and we do this by first constructing a reference graph with only those lines.

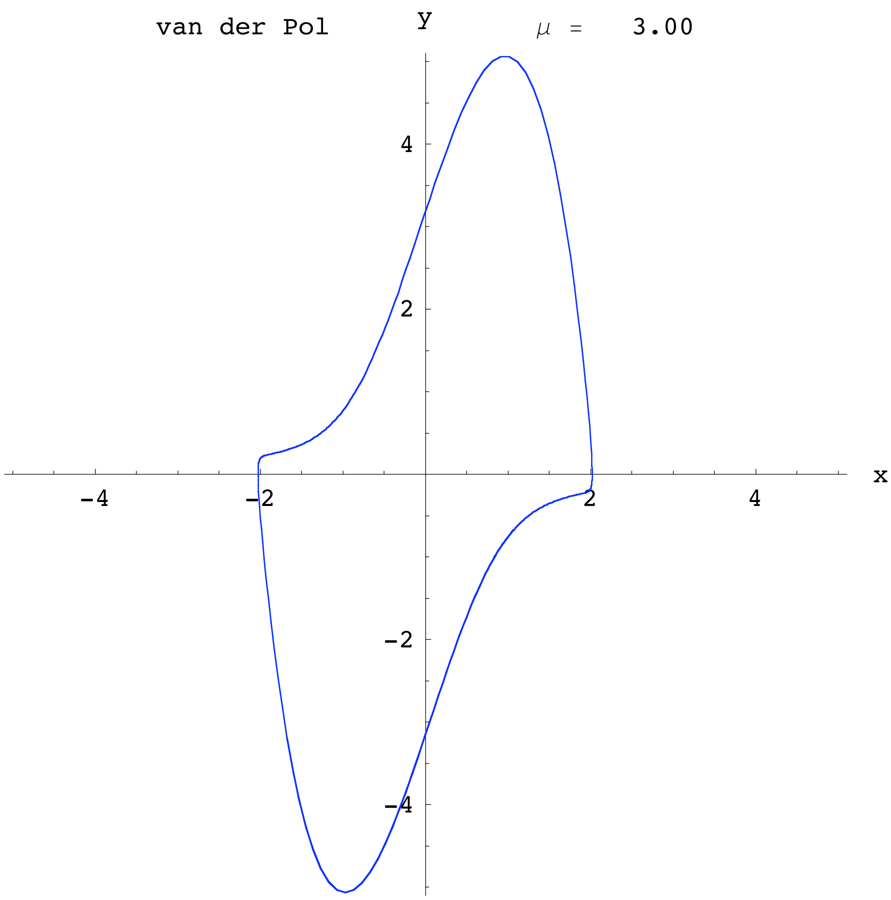
```
refgraph1 = ParametricPlot[{-1, t}, {t, -5.1, 5.1},
   PlotRange → plrange, AspectRatio → 1, ImageSize → imsize,
   PlotStyle → {Dashing[{0.02, 0.02}], RGBColor[1, 0, 0]},
   DisplayFunction → Identity];

refgraph2 = ParametricPlot[{1, t}, {t, -5.1, 5.1},
   PlotRange → plrange, AspectRatio → 1, ImageSize → imsize,
   PlotStyle → {Dashing[{0.02, 0.02}], RGBColor[1, 0, 0]},
   DisplayFunction → Identity];
```

Now we construct a sequence of graphs showing the motion of two phase points around the orbit. We will do this for $\mu = 3$.

```
parmval = {3};
```

We use the command lymcyc to get a pure limit cycle solution.

```
initvec = {1.5, 0}; t0 = 0.0; h = 0.02; nsteps = 500;
vansol = limcyc[initvec, t0, h, nsteps];
arrowflag = False;
refgraph3 = phaser[vansol];
```

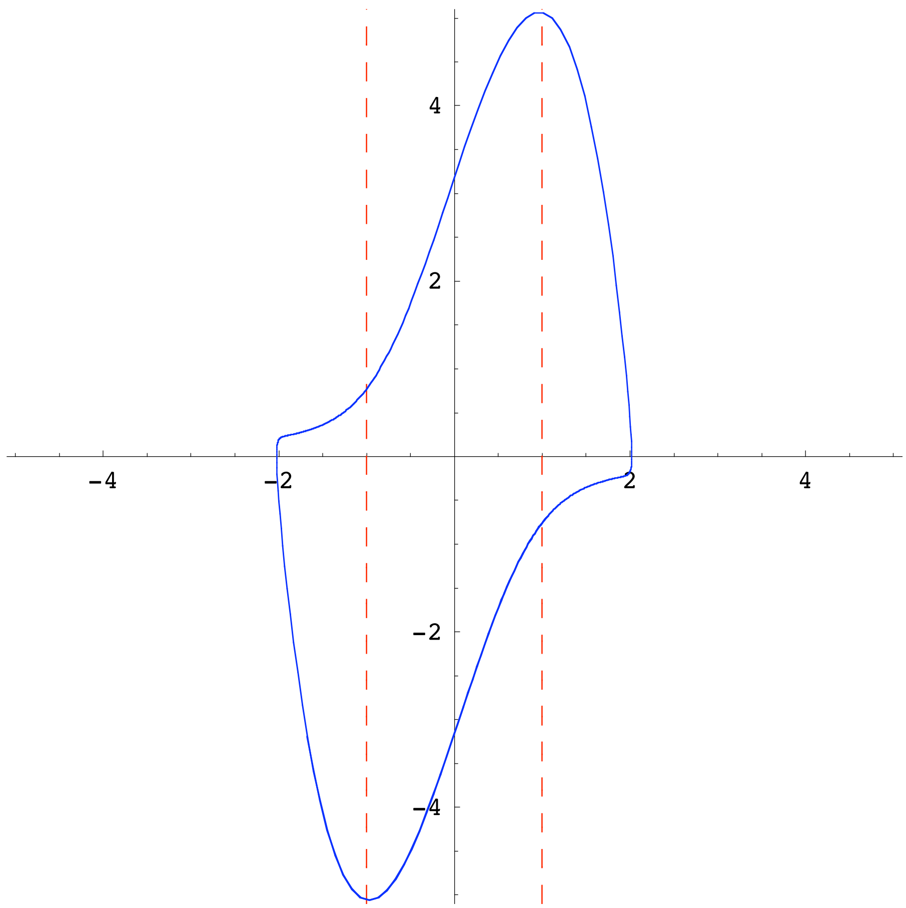



```
period[vansol]
```

```
8.86
```

We now construct a reference graph showing the limit cycle and the amplification boundaries at $x = \pm 1$.

```
refgraph = Show[{refgraph1, refgraph2, refgraph3},
   DisplayFunction → $DisplayFunction];
```



Now we construct the sequence showing the orbits traversed in real time. By scanning the solution vansol, we find two points that are separated by approximately half the cycle. These points are

```
initvec1 = {2.02330, 0.00253}; initvec2 = {-2.02324, 0.01618};
```

For h = 0.02, the number of time steps in one cycle is

```
period[vansol] / h
```

```
443.
```

We alter our time step slightly so that there are 600 steps in a cycle.

```
h = period[vansol] / 600
```

```
0.0147667
```

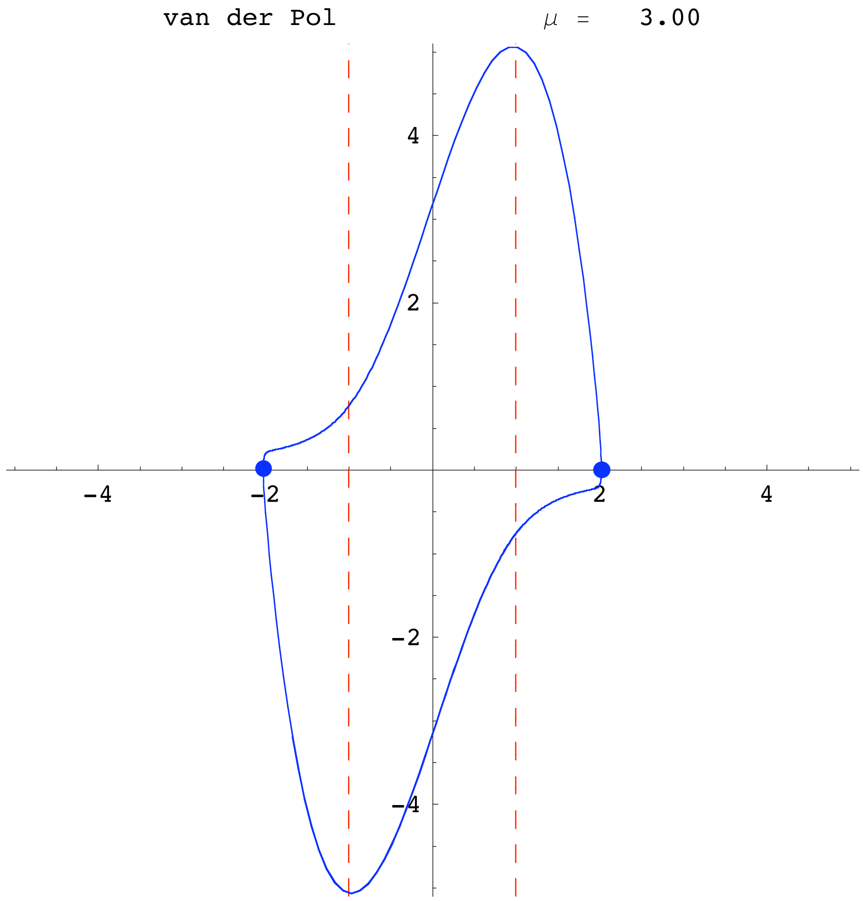
We will make 200 graphs in the sequence and they will cover one period, so we take 3 time steps between each picture. In the printed version of the notebook, only the first graph of the sequence is shown.

```
plrange = {{-5.1, 5.1}, {-5.1, 5.1}};
```

```
npics = 200; nstepic = 3; intlist = {initvec1, initvec2};
```

```
arrowflag = False;
```

```
phaseseq[intlist, t0, h, nstepic, npics, 1, 2, refgraph];
```



It is instructive to look at the time plot also. We first shift the time back to beginning at 0 by doing a timeshift of -10 on the solution. We check the shift by looking at the first point in the shifted solution.

```
vansolshift = timeshift[vansol, -10];
```

```
First[vansolshift]
```

```
{0., 1.07225, -0.675633}
```

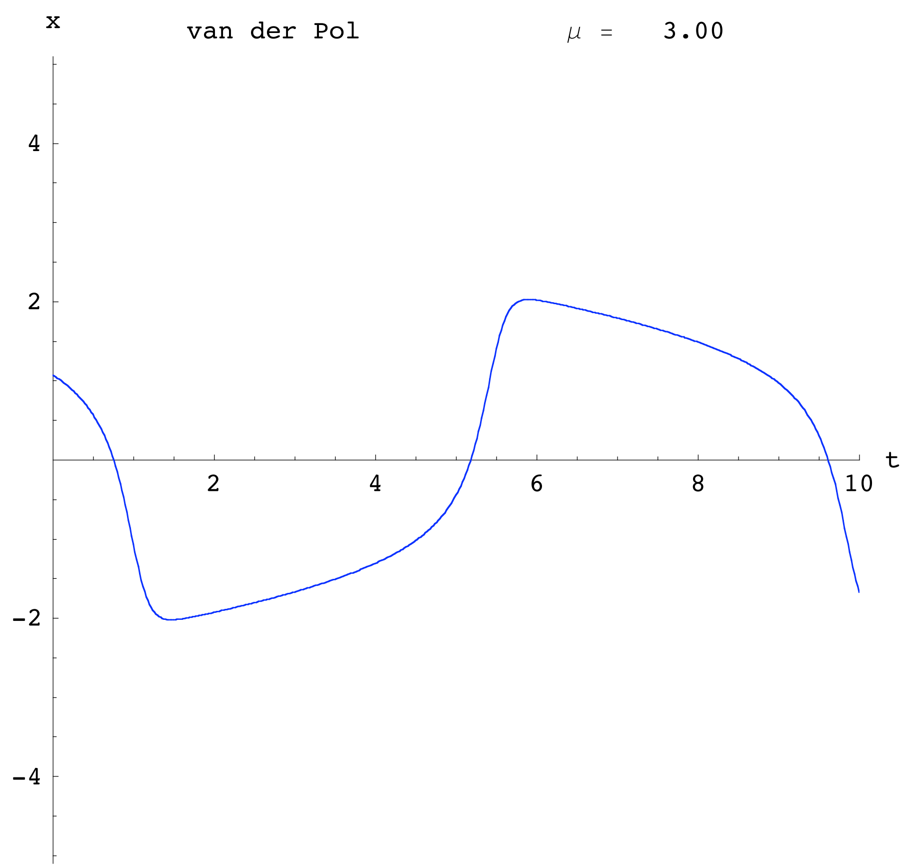


```
{0., 1.07225, -0.675633}
```



```
plrange = {{0, 10}, {-5.1, 5.1}};
```

```
timeplot[vansolshift, 1];
```



■ 7. Relaxation Oscillations

The orbit we have just seen is an example of a relaxation oscillation -- a kind of oscillation in which very slow segments are followed by very rapid segments. Let's see if we can use the van der Pol equation to understand why this happens. We rewrite the equation as

$$\mu (x^2 - 1) \dot{x} = -x - \ddot{x} \ . \tag{3}$$

When μ is large, either $\dot{x}$ is small or the term on the left hand side is large. If the term on the left is large, then it must be balanced by a large term on the right. The term x on the right is not large, so the only possibility is that $\ddot{x}$ is large. In a rough way, these arguments suggest that at any given point in the cycle, either the motion is very slow, in which case $\mu(x^2 -1)\dot{x}$ is small and is balanced by $-x$, or the motion is rapid, in which case $\mu(x^2 -1)\dot{x}$ is large and is balanced by $\ddot{x}$. If you go back and look at the movie again, you will see that the motion is consistent with these ideas. Here is a more detailed description. We start approximately at $x = 2$ and $y = 0$. The system is heavily damped, and there is a kind of creeping motion in which the damping force is balanced by the spring force, very much like a screen door closer. When x becomes less than 1, the damping changes to amplification. The system is rapidly accelerated and passes rapidly through the region from $x = 1$ to $x = -1$. When the system reaches the region to the left of $x = -1$, it is heavily damped, but it now has a lot of inertia. It is rapidly decelerated, and x is approximately -2 when the velocity falls to zero. Then the system creeps back toward $x = 0$, with damping and spring force in balance. When it reaches $x = -1$, the amplication begins again and the system is rapidly accelerated from $x = -1$ to $x = 1$. At $x = 1$, the system becomes heavily damped again, but inertia carries it out to about $x = 2$, where the velocity falls to zero, and the creeping motion begins again. It is possible to turn these qualititative ideas into a perturbation analysis of the system, based on the parameter μ being large. This is discussed in section 7.5 of Strogatz, and he gives there an approximate formula for the period τ for large μ, originally derived by Mary Cartwright in 1952. The formula is

$$\tau = (3 - 2 \ln 2)\mu + 2\alpha\mu^{-1/3} \quad , \qquad (4)$$

where $\alpha \approx 2.338$.