Lecture 10 Reprise and generalized forces

The Lagrangian

Holonomic constraints

Generalized coordinates

Nonholonomic constraints

Generalized forces

Euler-Lagrange equations

Hamilton’s equations

we haven’t done this, so let’s start with it
Generalized forces are based on the principle of virtual work

$$\delta W = F \cdot \delta \mathbf{r} + \tau \cdot \delta \Phi = Q_i \delta q^i$$

convert to time derivatives

$$\frac{\delta W}{\delta t} \Rightarrow \dot{W} \delta t, \quad Q_i \delta q^i \Rightarrow Q_i \dot{q}^i \delta t$$

$$\dot{W} \delta t = Q_i \dot{q}^i \delta t \Rightarrow \dot{W} = Q_i \dot{q}^i$$

$$Q_i = \frac{\partial \dot{W}}{\partial \dot{q}^i}$$

so what we need is $\dot{W}$
We need to look at the forces and torques applied to each link

\[ \dot{W} = F \cdot \dot{r} + \tau \cdot \dot{\Phi} \]

The translational part is obvious: \( F \cdot \dot{r} = F \cdot v \)

The rotational part

\[ \delta \Phi = \delta \phi k + \delta \theta I_1 + \delta \psi K_3 \]

\[ \delta \Phi = \dot{\phi} \delta t k + \dot{\theta} \delta t I_1 + \dot{\psi} \delta t K_3 = \omega \delta t = \Omega \delta t \]

\[ \dot{\Phi} = \omega \]

\[ \dot{W} = F \cdot v + \tau \cdot \omega \]
So my total rate of work will be

$$\dot{W} = \sum_{i=1}^{K} (\mathbf{F}_i \cdot \mathbf{v}_i + \tau_i \cdot \omega_i)$$

The velocities and rotations must be consistent with the constraints and everything must be expressed in inertial coordinates.

I can believe that this looks mysterious, but it is actually simple, which we can see by looking at some examples.

**REMEMBER THAT THESE FORCES AND TORQUES DO NOT INCLUDE THE POTENTIAL FORCES AND TORQUES**
This is the three link robot that we have seen before

It will have torques from the ground to link 1
from link 1 to link 2
from link 2 to link 3
The torque from the ground to link 1 $\tau_{01}k$

The torque from link 1 to link 2 $\tau_{12}I_1$

The torque from link 2 to link 3 $\tau_{23}I_2$

There is a force from the ground that balances the weight of the robot but it does no work because link 1 does not move up and down.
The torque on link 1 \( \tau_{01}k - \tau_{12}I_1 \)

The torque on link 2 \( \tau_{12}I_1 - \tau_{23}I_2 \)

The torque on link 3 \( \tau_{23}I_2 \)

These will give us the generalized forces as we’ll see shortly
There are a bunch of constraints affecting the rotation rates

\[ \phi_1 = 0 = \theta_1 \]

\[ \phi_2 = \psi_1 = \phi_3 \]

\[ \psi_2 = 0 = \psi_3 \]

There are also enough connectivity constraints to reduce the system to three degrees of freedom we’ll look at these shortly
All three $l$ vectors are equal to

$$\begin{bmatrix}
\cos \psi_1 \\
\sin \psi_1 \\
0
\end{bmatrix}$$

The rotation rates are

$$\begin{align*}
\omega_1 &= \begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_1 \end{bmatrix}, \\
\omega_2 &= \begin{bmatrix} \cos \psi_1 \dot{\theta}_2 \\ \sin \psi_1 \dot{\theta}_2 \\ \dot{\psi}_1 \end{bmatrix}, \\
\omega_3 &= \begin{bmatrix} \cos \psi_1 \dot{\theta}_3 \\ \sin \psi_1 \dot{\theta}_3 \\ \dot{\psi}_1 \end{bmatrix}
\end{align*}$$
Build the work terms for each link

\[ \omega_1 \cdot (\tau_{01} \mathbf{k} - \tau_{12} \mathbf{I}_1) = \tau_{01} \dot{\psi}_1 \]

\[ \omega_2 \cdot (\tau_{12} \mathbf{I}_1 - \tau_{23} \mathbf{I}_2) = (\tau_{12} - \tau_{23}) \dot{\theta}_2 \]

\[ \omega_3 \cdot \tau_{23} \mathbf{I}_2 = \tau_{23} \dot{\theta}_3 \]

so we have

\[ \dot{W} = \tau_{01} \dot{\psi}_1 + (\tau_{12} - \tau_{23}) \dot{\theta}_2 + \tau_{23} \dot{\theta}_3 \]
The three generalized coordinates

\[ \mathbf{q} = \begin{bmatrix} \psi_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \Rightarrow W = \tau_{01} \dot{q}_1 + (\tau_{12} - \tau_{23}) \dot{q}_2 + \tau_{23} \dot{q}_3 \]

Differentiate to get the generalized forces

\[ Q_1 = \tau_{01}, \quad Q_2 = (\tau_{12} - \tau_{23}), \quad Q_3 = \tau_{23} \]
We get a better idea of the compatibility issues by looking further.

Apply a horizontal force to the end of link 3
Let it be directed in the $-\mathbf{I}$ direction
We have to take the connectivity constraints in to account to handle this

We see intuitively that the only work is can do is to rotate the entire robot about \( k \).

It will contribute a torque to \( Q_1 \); can we get that using the rubric?

We have a force applied to link 3, and it is not at the center of mass, so there is an accompanying torque

\[
\tau_3 = \frac{1}{2} r_3 K_3 \times F = \frac{1}{2} r_3 K_3 \times I_1 F = \frac{1}{2} r_3 \sin \theta_3 F \dot{\psi}_1
\]
The connectivity constraint

\[ p_3 = p_1 + r_2 K_2 + \frac{1}{2} r_3 K_3 \Rightarrow v_3 = r_2 \dot{K}_2 + \frac{1}{2} r_3 \dot{K}_3 \]

The force piece of the work term is

\[ \dot{W}_F = F \cdot v_3 = FI_1 \cdot \left( r_2 \dot{K}_2 + \frac{1}{2} r_3 \dot{K}_3 \right) = \left( r_2 \sin \theta_2 + \frac{1}{2} r_3 \sin \theta_3 \right) F \hat{\psi}_1 \]

combine that with the torque term and we have

\[ \dot{W} = (r_2 \sin \theta_2 + r_3 \sin \theta_3) F \hat{\psi}_1 \]

for an additional term in accord with intuition

\[ Q_1 = (r_2 \sin \theta_2 + r_3 \sin \theta_3) F \]
We’ll look at this in general once again
The Lagrangian in physical variables

\[ L = \sum_{i=1}^{K} T_i - V_i \]

\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} A (\dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi)^2 \]

\[ + \frac{1}{2} B (-\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi)^2 + \frac{1}{2} C (\dot{\psi} + \dot{\phi} \cos \theta)^2 \]

We’ve mostly been doing gravity potentials
Simple holonomic constraints

Nonsimple holonomic constraints

These remove some number of variables from the problem

At this point you can assign the remaining variables to a set of generalized coordinates
Nonholonomic constraints: linear in the derivatives of $\mathbf{q}$

Write them in matrix form

$$C^i_j \dot{q}^j = 0 \iff \mathbf{C} \mathbf{\dot{q}} = 0$$

$\mathbf{C}$ is an $M \times N$ matrix —
number of constraints $\times$ number of generalized coordinates
Generalized forces

Identify the outside forces and torques on each link

Form \( \dot{W} \)

Then \( Q_i = \frac{\partial \dot{W}}{\partial \dot{q}^i} \)

Choose between Euler-Lagrange and Hamilton
Euler-Lagrange

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial L}{\partial q^i} + Q_i + \lambda_j C^j_i
\]

Convert to first order system

the first set of equations is really simple

\[
\dot{q}^i = u^i
\]

We need to develop the symbolic representation of the second set
Here’s the Lagrangian
we want its contribution to the left and right hand sides

$$L = \frac{1}{2} \dot{q}^i M_{ij} \dot{q}^j - V(q^k)$$

right hand side

$$\frac{\partial L}{\partial q^i} = \frac{1}{2} \dot{q}^m \frac{\partial M_{mn}}{\partial q^i} \dot{q}^n - \frac{\partial V}{\partial q^i}$$

substitute

$$\dot{q}^m = u^m, \quad \dot{q}^n = u^n$$

right hand side

$$\frac{\partial L}{\partial q^i} = \frac{1}{2} u^m \frac{\partial M_{mn}}{\partial q^i} u^n - \frac{\partial V}{\partial q^i}$$
left hand side

\[ \frac{\partial L}{\partial \dot{q}^i} = M_{ij} \dot{q}^j = M_{ij} u^j \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = M_{ij} \dot{u}^j + \frac{dM_{ij}}{dt} u^j \]

\[ M_{ij} \dot{u}^j + \frac{dM_{ij}}{dt} u^j = M_{ij} \dot{u}^j + \frac{\partial M_{ij}}{\partial q^k} u^j u^k \]

and we can combine the two and add back the constraint contribution and the generalized forces
\[ M_{ij} \dot{u}^j + \frac{\partial M_{ij}}{\partial q^k} u^j u^k = \frac{\partial L}{\partial q^i} + \lambda_j C_1^j + Q_i \]

\[ \frac{\partial L}{\partial q^i} = \frac{1}{2} u^m \frac{\partial M_{mn}}{\partial q^i} u^n - \frac{\partial V}{\partial q^i} \]

\[ M_{ij} \dot{u}^j + \frac{\partial M_{ij}}{\partial q^k} u^j u^k = \frac{1}{2} u^m \frac{\partial M_{mn}}{\partial q^i} u^n - \frac{\partial V}{\partial q^i} + \lambda_j C_1^j + Q_i \]
\[ M_{ij} \dot{u}^j + \frac{\partial M_{ij}}{\partial q^k} u^j u^k = \frac{1}{2} u^m \frac{\partial M_{mn}}{\partial q^i} u^n - \frac{\partial V}{\partial q^i} + \lambda_j C_1^j + Q_i \]

\[ M_{ij} \dot{u}^j = \frac{1}{2} u^m \frac{\partial M_{mn}}{\partial q^i} u^n - \frac{\partial M_{ij}}{\partial q^k} u^j u^k - \frac{\partial V}{\partial q^i} + \lambda_j C_1^j + Q_i \]

\[ \dot{q}^i = u^i \]

and we can (in principle) solve for the individual components of \( u \)

\[ \dot{u}^j = M^{ki} \left( \frac{1}{2} u^m \frac{\partial M_{mn}}{\partial q^i} u^n - \frac{\partial M_{ij}}{\partial q^k} u^j u^k - \frac{\partial V}{\partial q^i} + \lambda_j C_1^j + Q_i \right) \]
We did Hamilton the last time

I have the pair of sets of odes

\[ \dot{q}^j = M^{ji} p_i \]

\[ \dot{p}_i = \frac{\partial L}{\partial q_i} + \lambda_j C^j_i + Q_i \]

where all the q dots in the second set have been replaced by their expressions in terms of p
We can write this out in detail, although it looks pretty awful

\[ L = \frac{1}{2} \dot{q}^i M_{ij} \dot{q}^j - V(q^k) \]

right hand side

\[ \frac{\partial L}{\partial q^i} = \frac{1}{2} \dot{q}^m \frac{\partial M_{mn}}{\partial q^i} \dot{q}^n - \frac{\partial V}{\partial q^i} \]

substitute

\[ \dot{q}^m = \bar{M}^{mr} p_r, \quad \dot{q}^n = \bar{M}^{ns} p_s \]

right hand side

\[ \frac{\partial L}{\partial q^i} = \frac{1}{2} \bar{M}^{mr} p_r \frac{\partial M_{mn}}{\partial q^i} \bar{M}^{ns} p_s - \frac{\partial V}{\partial q^i} \]
\[
\dot{p}_i = \frac{\partial L}{\partial q^i} + \lambda_j C_j^i + Q_i \\
\frac{\partial L}{\partial q^i} = \frac{1}{2} \overline{M}^{mr} p_r \frac{\partial M_{mn}}{\partial q^i} \overline{M}^{ns} p_s - \frac{\partial V}{\partial q^i} \\
\dot{p}_i = \frac{1}{2} \overline{M}^{mr} p_r \frac{\partial M_{mn}}{\partial q^i} \overline{M}^{ns} p_s - \frac{\partial V}{\partial q^i} + \lambda_j C_j^i + Q_i \\
\dot{q}^i = \overline{M}^{ij} p_j
\]
Both of these are of the same general form: a form that we will see in the other methods we will address.

\[
\begin{align*}
\dot{q}^i &= M^{ij} p_j \\
\dot{p}_i &= \frac{1}{2} M^{mr} \dot{p}_r \frac{\partial M_{mn}}{\partial q^i} M^{ns} p_s - \frac{\partial V}{\partial q^i} + \lambda_j C_1^j + Q_i
\end{align*}
\]
What you have to do in general — whether you use Euler-Lagrange or Hamilton

Set up the physical Lagrangian

Put in the simple holonomic constraints

Put in the nonsimple holonomic constraints

Define the generalized coordinates

Find the nonholonomic constraints and the constraint matrix

Define the Lagrange multipliers

Find the rate of work and the generalized forces
For Hamilton’s equations

Define the momentum and use it to eliminate \( \dot{q}^i = \overline{M}^{ij} p_j \)

(These are half the differential equations)

The other half are the momentum equations

\[
\dot{p}_i = \frac{1}{2} \overline{M}^{mr} p_r \frac{\partial M_{mn}}{\partial q^i} \overline{M}^{ns} p_s - \frac{\partial V}{\partial q^i} + \lambda_j C^j_i + Q_i
\]
Solve some of the momentum equations for the Lagrange multipliers

Combine the remaining momentum equations with the differentiated constraints

These are the new momentum equations

Solve the new momentum equations simultaneously with $\dot{q}^i = \overline{M}^{ij} p_j$
An example

link 1, the frame
link 2, the fork
link 3, the rear wheel
link 4, the front wheel
The (physical) Lagrangian

There are four identical Lagrangians — we’ve seen them before
I’m not going to write them out

I am going to hold this system erect, so that gravity does nothing
I can set $g = 0$
Simple holonomic constraints

I want the system to be erect
Let the frame and fork rotate only about the vertical

\[ \phi_1 = 0 = \phi_2, \quad \theta_1 = 0 = \theta_2 \quad \text{minus 4} \]

Define the difference between their \( k = K \) rotations

\[ \psi_2 = \psi_1 + \alpha \quad \text{no change} \]
Simple holonomic constraints

The heights of all the links are fixed

$$z_1 = h_1, \quad z_2 = h_2, \quad z_3 = r_w = z_4 \quad \text{minus 4}$$
Simple holonomic constraints

I want the wheels to be upright

\[ \theta_3 = -\frac{\pi}{2} = \theta_4 \quad \text{minus 2} \]

The rear wheel points in the same direction as the frame
the front wheel in the same direction as the fork

\[ \phi_3 = \psi_1, \quad \phi_4 = \psi_1 + \alpha \quad \text{minus 2} \]

The horizontal locations of the centers of mass are connected
the front connection is simple

\[ x_4 = x_3, \quad y_4 = y_3 \quad \text{minus 2} \]
Nonsimple holonomic constraints

The other horizontal locations of the centers of mass are also connected

\[ x_3 = -l_1 \cos \psi_1, \quad y_3 = -l_1 \sin \psi_1 \]
\[ x_2 = l_2 \cos \psi_1, \quad y_2 = l_2 \sin \psi_2 \]

At this point we have introduced eighteen constraints, leaving six variables.
Assign the generalized coordinates

\[ q = \begin{bmatrix} x_1 \\ y_1 \\ \psi_1 \\ \alpha \\ \psi_3 \\ \psi_4 \end{bmatrix} \]

We still have rolling constraints
Nonholonomic constraints

rolling constraints

\[ \mathbf{v}_3 = \omega_3 \times \mathbf{r}_3, \quad \mathbf{v}_4 = \omega_4 \times \mathbf{r}_4 \]

\[ \mathbf{r}_3 = \mathbf{k} = \mathbf{r}_4 \]

vertical components are already in place, so there are but four of these

\[
\begin{align*}
\dot{x}_3 - r_w \psi_3 \cos \phi_3 &= 0 = \dot{y}_3 - r_w \psi_3 \sin \phi_3 \\
\dot{x}_4 - r_w \psi_4 \cos \phi_4 &= 0 = \dot{y}_4 - r_w \psi_4 \sin \phi_4
\end{align*}
\]

constraint matrix

\[
\mathbf{C} = \begin{bmatrix}
1 & 0 & l_2 \sin q^3 & 0 & -r_w \cos q^3 & 0 \\
0 & 1 & -l_2 \cos q^3 & 0 & -r_w \sin q^3 & 0 \\
1 & 0 & -l_1 \sin q^3 & 0 & 0 & -r_w \cos(q^3 + q^4) \\
0 & 1 & l_1 \cos q^3 & 0 & 0 & -r_w \sin(q^3 + q^4)
\end{bmatrix}
\]
Nonholonomic constraints

in physical coordinates

\[ C = \begin{bmatrix}
1 & 0 & l_2 \sin \psi_1 & 0 & -r_w \cos \psi_1 & 0 \\
0 & 1 & -l_2 \cos \psi_1 & 0 & -r_w \sin \psi_1 & 0 \\
1 & 0 & -l_1 \sin \psi_1 & 0 & 0 & -r_w \cos (\psi_1 + \alpha) \\
0 & 1 & l_1 \cos \psi_1 & 0 & 0 & -r_w \sin (\psi_1 + \alpha)
\end{bmatrix} \]

There will be four Lagrange multipliers

We’ll add \( \lambda_j C_i^j \) to the momentum equations
Generalized forces

There is torque from the frame to the fork \( \tau_{12} \mathbf{k} \)

and from the frame to the rear wheel \( \tau_{13} \mathbf{K}_3 \)

The rate of work will be

\[
\dot{W} = \omega_1 \cdot (-\tau_{12} \mathbf{k} - \tau_{13} \mathbf{K}_3) + \omega_2 \cdot \tau_{12} \mathbf{k} + \omega_3 \cdot \tau_{13} \mathbf{K}_3
\]

from which we can find

\[
\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & \tau_{12} & \tau_{13} & 0 \end{bmatrix}
\]
At this point we have the structure of the problem

It’s time to go to Mathematica to see how to do this in practice

And to see how to convert this to numerical form for integration

And some simple examples