Lecture 5: Constraints I

Constraints reduce the number of degrees of freedom of a mechanism

holonomic constraints

  external constraints (bead on a wire)

  internal constraints (connectivity constraints)

orientation constraints
Holonomic constraints can be written in terms of the coordinates

Simple holonomic constraints are linear in the coordinates

Nonsimple holonomic constraints are nonlinear in the coordinates

(I will address nonholonomic constraints in Lecture 7.)
Bead on a wire

I call it a bead, and I’ll treat it as a point
— three degrees of freedom
A little digression on the properties of curves in space
the tangent vector

\[ \lambda = \frac{dp}{ds} \]

This is the unit tangent vector
we’ll do a lot with nonunit tangent vectors
Bead on a wire

Curve is defined by a parameterization

\[ x = x(s), \quad y = y(s), \quad z = z(s) \quad \text{arclength} \]

\[ x = x(\chi), \quad y = y(\chi), \quad z = z(\chi) \quad \text{general parameter} \]

\[ \lambda = \begin{bmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \\ \frac{dz}{ds} \end{bmatrix} = \frac{1}{ds} \begin{bmatrix} \frac{dx}{d\chi} \\ \frac{dy}{d\chi} \\ \frac{dz}{d\chi} \end{bmatrix} \]
Bead on a wire

Kinetic energy

\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m (x''^2 + y''^2 + z''^2) \left( \frac{d\chi}{dt} \right)^2 \]

add in the potential energy and write the Lagrangian

\[ L = \frac{1}{2} m \left( x'(\chi)^2 + y'(\chi)^2 + z'(\chi)^2 \right) \left( \frac{d\chi}{dt} \right)^2 - mgz(\chi) \]
Bead on a wire

The only variable here is $\chi$ (a proxy for $s$), so we build one EL equation

$$\frac{\partial L}{\partial \dot{\chi}} = m\left(x'(\chi)^2 + y'(\chi)^2 + z'(\chi)^2\right)\dot{\chi}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\chi}}\right) = m\left(x'(\chi)^2 + y'(\chi)^2 + z'(\chi)^2\right)\ddot{\chi} + m\frac{d}{dt}\left(x'(\chi)^2 + y'(\chi)^2 + z'(\chi)^2\right)\dot{\chi}$$

$$\frac{\partial L}{\partial \chi} = m\left(x''(\chi) + y''(\chi) + z''(\chi)\right)\dot{\chi}^2 - mgz'(\chi)$$
Bead on a wire

The Euler-Lagrange equation

\[ m \left( x'(\chi)^2 + y'(\chi)^2 + z'(\chi)^2 \right) \ddot{\chi} + m (x'' + y'' + z'') \dot{\chi}^2 - mgz' = 0 \]
Bead on a wire

We can solve this analytically in the simple case where

\[ x'(\chi)^2 + y'(\chi)^2 + z'(\chi)^2 \]

is a constant.

The helix is such a case

\[ x = a\cos(\chi), \quad y = a\sin \chi, \quad z = -h\chi \]

\[ x'(\chi)^2 + y'(\chi)^2 + z'(\chi)^2 = a^2 \sin^2 \chi + a^2 \cos^2 \chi + h^2 = a^2 + h^2 \]
Bead on a wire

\[
\frac{\partial L}{\partial \dot{\chi}} = m\left(x'(\chi)^2 + y'(\chi)^2 + z'(\chi)^2\right)\ddot{\chi} = m(a^2 + h^2)\ddot{\chi}
\]

so the Euler-Lagrange equation is

\[
m(1 + h^2)\ddot{\chi} - mgh = 0
\]

which is easily integrated to give

\[
\chi = \chi_0 + \dot{\chi}_0 t + g \frac{ht^2}{2(a^2 + h^2)}
\]
Bead on a wire

From which we obtain

\[ x = a \cos \left( \chi_0 + \dot{\chi}_0 t + g \frac{ht^2}{2(a^2 + h^2)} \right) \]

\[ y = a \sin \left( \chi_0 + \dot{\chi}_0 t + g \frac{ht^2}{2(a^2 + h^2)} \right) \]

\[ z = h\chi_0 + h\dot{\chi}_0 t + g \frac{h^2 t^2}{2(a^2 + h^2)} \]
Bead on a wire

The force on the bead from the wire

\[ f_x = m\ddot{x} = -am\cos(\Phi)\left(\ddot{x}_0 + \dot{\chi}_0 + g\frac{ht}{(a^2 + h^2)}\right)^2 - am\sin(\Phi)\frac{gh}{(a^2 + h^2)} \]

\[ f_y = m\ddot{y} = -am\sin(\Phi)\left(\ddot{y}_0 + \dot{\chi}_0 + g\frac{ht}{(a^2 + h^2)}\right)^2 + am\cos(\Phi)\frac{gh}{(a^2 + h^2)} \]

\[ f_z = m\ddot{z} - mg = -\frac{mga^2}{(a^2 + h^2)} \]

\[ \Phi = \chi_0 + \dot{\chi}_0 t + g\frac{ht^2}{2(a^2 + h^2)} \]
Bead on a (complicated) wire

wire profile

\[ \{ \cos(t) \cos(3t), \sin(t) \cos(3t), \frac{1}{2} \cos(10t) + \frac{1}{2} \} \]
Bead on a (complicated) wire

This one has to be done numerically

\[ T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \]

\[ V = mgz \]

Constraints are from the previous slide

\{ \cos(t) \cos(3t), \sin(t) \cos(3t), \frac{1}{2} \cos(10t) + \frac{1}{2} \}
Bead on a (complicated) wire

The constrained Lagrangian

\[ L = \frac{1}{4} m(25 \cos(20\chi) + 8 \cos(6\chi) - 35) \dot{\chi}^2 - \frac{1}{2} mg(\cos^2(5\chi)) \]

The Euler-Lagrange equation

\[ (25 \cos(20\chi) + 8 \cos(6\chi) - 35) \ddot{\chi} + \frac{1}{2} (500 \sin(20\chi) + 48 \sin(6\chi)) \dot{\chi}^2 - 20 g \cos(5\chi) \sin(5\chi) = 0 \]

Make two first order equations out of this
Bead on a (complicated) wire

\[ \dot{\chi} = u \]

\[ u = \frac{2\left(\left(125\sin(20\chi) + 12\sin(6\chi)\right)u^2 - 5g\cos(5\chi)\sin(5\chi)\right)}{25\cos(20\chi) + 8\cos(6\chi) - 35} \]

The maximum height of the track is unity at \( \chi = 0 \)

Starting from rest below unity leads to an oscillation

Starting at \( \chi = 0 \) with no motion gives (unstable) equilibrium

Add motion and we get a roller coaster-like motion

Let’s look at the Mathematica code
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The Lagrangian, the Euler-Lagrange equations and the partition into two equations

```
In[15]:= L = Collect[L, \(\chi t\), Simplify]
Out[15]= (-3 g Cos[5 \(\chi\)]^2 - \(\frac{3}{4}\) \(\chi\) t^2 (-35 + 8 Cos[6 \(\chi\)] + 25 Cos[20 \(\chi\)]))

In[16]:= D[L, \(\chi t\)];
D[%, \(\chi\)] \(\chi\) t + D[%, \(\chi t\)] \(\chi\) t t;
lhs = % - D[L, \(\chi\)]
Out[18]= \(-\frac{3}{2}\) \(\chi\) t t (-35 + 8 Cos[6 \(\chi\)] + 25 Cos[20 \(\chi\)]) -
30 g Cos[5 \(\chi\)] Sin[5 \(\chi\)] - \(\frac{3}{4}\) \(\chi\) t^2 (-48 Sin[6 \(\chi\)] - 500 Sin[20 \(\chi\)])

In[19]:= Solve[lhs == 0, \(\chi\) t t];
u_t = Simplify[\(\chi\) t t /. %[[1]]]
Out[20]= \(\frac{2 \left(12 \chi t^2 \sin[6 \chi] - 5 g \sin[10 \chi] + 125 \chi t^2 \sin[20 \chi]\right)}{-35 + 8 Cos[6 \chi] + 25 Cos[20 \chi]}\)
```
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The odes, the initial conditions and a space for the answer

\begin{align*}
\text{In}[21]:& \quad \text{ode1} = x'[t] = u[t]; \\
& \text{ode2} = u'[t] = ut\. \{x \rightarrow x[t], xt \rightarrow x'[t]\}
\end{align*}

\begin{align*}
\text{Out}[22]:& \quad u'[t] = \frac{2 \left( -5 g \sin[10 x[t]] + 12 \sin[6 x[t]] x'[t]^2 + 125 \sin[20 x[t]] x'[t]^2 \right)}{-35 + 8 \cos[6 x[t]] + 25 \cos[20 x[t]]}
\end{align*}

\begin{align*}
\text{In}[23]:& \quad \text{ics} = \{x[0] = x0, u[0] = u0\} \\
& \text{ans} = \{x[t], u[t]\}
\end{align*}

\begin{align*}
\text{Out}[23]:& \quad \{x[0] = x0, u[0] = u0\} \\
\text{Out}[24]:& \quad \{x[t], u[t]\}
\end{align*}

\begin{align*}
\text{In}[25]:& \quad \text{odes} = \{\text{ode1}, \text{ode2}\};
\end{align*}
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The solution procedure and the answers

```math
\chi_0 = 0; u_0 = 0.1; g = 9.81; tf = 8 \pi;
\text{soln} = \text{NDSolve}[[\text{odes, ics}, \text{ans, \{t, 0, tf\}}];
\text{nX} = \text{soln}[[1, 1, 2]];
\text{nu} = \text{soln}[[1, 2, 2]];
\text{nx} = x /\to \text{nX};
\text{ny} = y /\to \text{nX};
\text{nz} = z /\to \text{nX};
\text{nxt} = \text{xt} /\to \{x \to \text{nX}, \text{xt} \to \text{nu}\};
\text{nyt} = \text{yt} /\to \{x \to \text{nX}, \text{xt} \to \text{nu}\};
\text{nzt} = \text{zt} /\to \{x \to \text{nX}, \text{xt} \to \text{nu}\};
\text{speed} = \text{Sqrt}[\text{nxt}^2 + \text{nyt}^2 + \text{nzt}^2];
\text{energy} = 1/2 \text{m speed}^2 + \text{mg nz};
\text{Plot}[\text{speed, \{t, 0, tf\}}]
\text{Plot}[\text{energy, \{t, 0, tf\}}]
\text{Plot}[\text{nz, \{t, 0, tf\}}]
\text{ParametricPlot3D}[\{\text{nx, ny, nz}, \{t, 0, tf\}}]
\text{ParametricPlot}[\{\text{nz, speed}, \{t, 0, tf\}}]
```
Bead on a (complicated) wire

RESULTS

Start from $\chi = 0$ and $u = 0.1$ and let $0 \leq t \leq 8\pi$
This gives several runs around the track

We can look at speed vs. time
height vs. time
energy vs. time (conserved)
Bead on a (complicated) wire

speed
Bead on a (complicated) wire

height
Bead on a (complicated) wire

energy
Bead on a (complicated) wire

speed vs. height

initial speed is 0.1
Link on a wire

I will eventually specialize to an axisymmetric link on the helical wire, but we can say some general things first.

I will align the $\mathbf{K}$ axis of the link with the local curve.
Link on a wire

I suppose the link to be small enough that I can neglect the curvature of the wire locally, so I can still impose the position constraints

\[ x = x(\chi), \quad y = y(\chi), \quad z = z(\chi) \]

But now I have an orientation constraint

\[ \mathbf{K} = \lambda \Rightarrow \sin \theta \sin \phi \mathbf{i} - \sin \theta \cos \phi \mathbf{j} + \cos \theta \mathbf{k} \propto x' \mathbf{i} + y' \mathbf{j} + z' \mathbf{k} \]

\[ \tan \phi = -\frac{x'}{y'} \Rightarrow \sin \theta \frac{x'}{\sqrt{x'^2 + y'^2}} \mathbf{i} - \sin \theta \frac{y'}{\sqrt{x'^2 + y'^2}} \mathbf{j} + \cos \theta \mathbf{k} \propto x' \mathbf{i} + y' \mathbf{j} + z' \mathbf{k} \]
Link on a wire

divide lhs by \( \sin \theta \)

\[
\frac{x'}{\sqrt{x'^2 + y'^2}} \mathbf{i} - \frac{y'}{\sqrt{x'^2 + y'^2}} \mathbf{j} + \cot \theta \mathbf{k} \propto \mathbf{r}' = x' \mathbf{i} + y' \mathbf{j} + z' \mathbf{k}
\]

\[
\cot \theta = \frac{z'}{\sqrt{x'^2 + y'^2}}
\]

We have determined two Euler angles: \( \phi \) and \( \theta \)

We are left with \( \psi \), and the system has two degrees of freedom
Link on a wire

Doing this in general is really difficult and it will obscure what is going on in a sea of algebra so let’s stick to the helix

\[ x = a \cos(\chi), \quad y = a \sin(\chi), \quad z = -h \chi \]

\[ x' = -a \sin(\chi), \quad y' = a \cos(\chi), \quad z' = -h \]

\[ \mathbf{K} = \sin \theta \sin \phi \mathbf{i} - \sin \theta \cos \phi \mathbf{j} + \cos \theta \mathbf{k} \]

\[ \mathbf{\lambda} \propto -a \sin \chi \mathbf{i} + a \cos \chi \mathbf{j} - h \mathbf{k} \]
Link on a wire

Start by letting $\phi = \chi$, which is almost intuitive

$$K = \sin \theta \sin \chi \mathbf{i} - \sin \theta \cos \chi \mathbf{j} + \cos \theta \mathbf{k}$$

$$\lambda \propto -a \sin \chi \mathbf{i} + a \cos \chi \mathbf{j} - h \mathbf{k}$$

$\sin \theta$ needs to be proportional to $-a$
$\cos \theta$ needs to be proportional to $-h$

$$\sin \theta = -\frac{a}{\sqrt{a^2 + h^2}}, \quad \cos \theta = -\frac{h}{\sqrt{a^2 + h^2}}$$

$$\theta = -\sin^{-1}\left(\frac{h}{\sqrt{a^2 + h^2}}\right) - \frac{\pi}{2}$$
Link on a wire

Two degrees of freedom — $\chi$ and $\psi$ — as we noted before

If the link is axisymmetric this problem has a closed from solution!

I’m going to leave this as a problem for you
We’ve been looking at external constraints on a single link

I’d like to start on internal constraints connecting links

I will split these into orientation constraints and connectivity constraints

I will also split them into simple and nonsimple constraints

Orientation constraints are often simple
   Connectivity constraints are usually nonsimple
Some notation

I’m going to work, for now, with fairly symmetric bodies
their centers of mass are also their “actual center”

rectangular solids
cylinders
everipsoids

I will use body coordinates parallel to the principal moments

I will usually (at least for now) connect bodies at their ends
and suppose the ends to be in the \( K \) direction
I suppose the semiaxes to be $a$, $b$ and $c$ in the $I$, $J$ and $K$ directions.

Here $b = a$.

The connectivity of these two links relates the center of mass of link two to the center of mass of link one:

$$\mathbf{r}_2 = \mathbf{r}_1 + c_1\mathbf{K}_1 + c_2\mathbf{K}_2$$

This relates $x_2$, $y_2$ and $z_2$ to the coordinates of link 1 and all six Euler angles.

It’s a connectivity constraint and it is not simple — by which I mean not linear.
\[ r_2 = r_1 + c_1 (\sin \theta_1 \sin \phi_1 i - \sin \theta_1 \cos \phi_1 j + \cos \theta_1 k) + c_2 (\sin \theta_2 \sin \phi_2 i - \sin \theta_2 \cos \phi_2 j + \cos \theta_2 k) \]

\[
x_2 = x_1 + c_1 \sin \theta_1 \sin \phi_1 + c_2 \sin \theta_2 \sin \phi_2
\]

\[
y_2 = y_1 - c_1 \sin \theta_1 \cos \phi_1 - c_2 \sin \theta_2 \cos \phi_2
\]

\[
z_2 = z_1 + c_1 \cos \theta_1 + c_2 \cos \theta_2
\]

The system has nine degrees of freedom
   It started with twelve and we took three out

You can see that substituting this into the Lagrangian will lead to quite a mess

It gets worse as we add links

We’re going to learn cool ways to deal with this, but not just yet
We can further constrain this system by attaching the first link to the ground at the origin of the inertial system

\[ \mathbf{r}_1 = c_1 \mathbf{K}_1, \quad \mathbf{r}_2 = 2c_1 \mathbf{K}_1 + c_2 \mathbf{K}_2 \]

\[ x_1 = c_1 \sin \theta_1 \sin \phi_1, \quad x_2 = 2c_1 \sin \theta_1 \sin \phi_1 + c_2 \sin \theta_2 \sin \phi_2 \]
\[ y_1 = -c_1 \sin \theta_1 \cos \phi_1, \quad y_2 = -2c_1 \sin \theta_1 \cos \phi_1 - c_2 \sin \theta_2 \cos \phi_2 \]
\[ z_1 = c_1 \cos \theta_1, \quad z_2 = 2c_1 \cos \theta_1 + c_2 \cos \theta_2 \]

and this system now has six degrees of freedom
Suppose the two links we looked at to be connected by a hinge instead of a magic spherical joint.

Here $\mathbf{I}$ denotes the common $\mathbf{I}$ vector for both links, the direction of the hinge pin.
\[ \mathbf{I}_1 = (\cos \phi_1 \cos \psi_1 - \sin \phi_1 \cos \theta_1 \sin \psi_1 ) \mathbf{i} + (\sin \phi_1 \cos \psi_1 + \cos \phi_1 \cos \theta_1 \sin \psi_1 ) \mathbf{j} + \sin \theta_1 \sin \psi_1 \mathbf{k} \]

\[ \mathbf{I}_2 = (\cos \phi_2 \cos \psi_2 - \sin \phi_2 \cos \theta_2 \sin \psi_2 ) \mathbf{i} + (\sin \phi_2 \cos \psi_2 + \cos \phi_2 \cos \theta_2 \sin \psi_2 ) \mathbf{j} + \sin \theta_2 \sin \psi_2 \mathbf{k} \]

and these will be equal if

\[ \phi_2 = \phi_1, \quad \psi_2 = 0 = \psi_1 \]

(It is possible to have a more complicated hinge relation
and we may get to that near the end of the course
or we may not.)
Think about this operationally

We rotate both of these through the same $\phi$ angle

Then we rotate each about a different $\theta$ angle

We do not rotate either about its $\psi$ angle
initial position
first rotation
second rotations