Lecture 9 Hamilton’s Equations

Informal derivation

conjugate momentum

cyclic coordinates

Applications/examples
derivation

Start with the Euler-Lagrange equations

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = \lambda_j C^j_1 + Q_i \]

Define the \textbf{conjugate momentum}

\[ p_i = \frac{\partial L}{\partial \dot{q}^i} \]

The Euler-Lagrange equations can be rewritten as

\[ \dot{p}_i = \frac{\partial L}{\partial q^i} + \lambda_j C^j_1 + Q_i \]
If we are to set up a pair of sets of odes, we need to eliminate $\dot{q}^i$

We can write the Lagrangian

$$L = \frac{1}{2} \dot{q}^i M_{ij} \dot{q}^j - V(q^k)$$

from which $p_i = M_{ij} \dot{q}^j$

$M$ is positive definite (coming from the kinetic energy), so

$$\dot{q}^j = M^{ji} p_i$$
I have the pair of sets of odes

\[ \dot{q}_j = M_{ji} p_i \]

\[ \dot{p}_i = \frac{\partial L}{\partial q_i} + \lambda_j C_1^j + Q_i \]

where all the q dots in the second set have been replaced by their expressions in terms of p
We can write this out in detail, although it looks pretty awful

\[ L = \frac{1}{2} \dot{q}^i M_{ij} \dot{q}^j - V(q^k) \]

right hand side

\[ \frac{\partial L}{\partial q^i} = \frac{1}{2} \dot{q}^m \frac{\partial M_{mn}}{\partial q^i} \dot{q}^n - \frac{\partial V}{\partial q^i} \]

substitute

\[ \dot{q}^m = \bar{M}^{mr} p_r, \quad \dot{q}^n = \bar{M}^{ns} p_s \]

right hand side

\[ \frac{\partial L}{\partial q^i} = \frac{1}{2} \bar{M}^{mr} p_r \frac{\partial M_{mn}}{\partial q^i} \bar{M}^{ns} p_s - \frac{\partial V}{\partial q^i} \]
\[ \dot{p}_i = \frac{\partial L}{\partial q_i} + \lambda_j C^j_i + Q_i \]

\[ \frac{\partial L}{\partial q^i} = \frac{1}{2} M^{mr} p_r \frac{\partial M_{mn}}{\partial q^i} M^{ns} p_s - \frac{\partial V}{\partial q^i} \]

We will never do it this way!

I will give you a recipe as soon as we know what cyclic coordinates are
??
cyclic coordinates

Given

\[ \dot{p}_i = \frac{\partial L}{\partial q^i} + \lambda_j C^j_i + Q_i \]

If \( Q_i \) and the constraints are both zero (free falling brick, say) we have

\[ \dot{p}_i = \frac{\partial L}{\partial q^i} \]

and if \( L \) does not depend on \( q^i \), then \( p_i \) is constant!

**Conservation of conjugate momentum**

\( q^i \) is a cyclic coordinate \( \dot{p}_i = 0 \)
YOU NEED TO REMEMBER THAT EXTERNAL FORCES AND/OR CONSTRAINTS CAN MAKE CYCLIC COORDINATES NON-CYCLIC!
What can we say about an unforced single link in general?

\[
L = \frac{1}{2} \left( A(\dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi)^2 + B(-\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi)^2 + C(\dot{\psi} + \dot{\phi} \cos \theta)^2 \right) \\
+ \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz
\]

We see that \(x\) and \(y\) are cyclic (no explicit \(x\) or \(y\) in \(L\))

Conservation of linear momentum in \(x\) and \(y\) directions

\[
p_x = p_1 = m \dot{x}, \quad p_y = p_2 = m \dot{y}
\]

We see that \(\phi\) is also cyclic (no explicit \(\phi\) in \(L\))
The conserved conjugate momentum is

\[ p_\phi = p_4 = \left( (A \sin^2 \psi + B \cos^2 \psi) \sin^2 \theta + C \cos^2 \theta \right) \dot{\phi} + (A - B) \sin \theta \sin \psi \cos \psi \dot{\theta} + C \cos \theta \dot{\psi} \]

Does it mean anything physically?!

**the angular momentum about the k axis**

as I will now show you
The angular momentum in body coordinates is

\[ \mathbf{l} = A(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\mathbf{I}_3 + B(-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi)\mathbf{J}_3 + C(\dot{\psi} + \dot{\phi}\cos\theta)I_{zz}\mathbf{K}_3 \]

The body axes are (from Lecture 3)

\[ \mathbf{I}_3 = \cos\psi(\cos\phi\mathbf{I}_0 + \sin\phi\mathbf{J}_0) + \sin\psi(\cos\theta(-\sin\phi\mathbf{I}_0 + \cos\phi\mathbf{J}_0) + \sin\theta\mathbf{K}_0) \]
\[ \mathbf{J}_3 = -\sin\psi(\cos\phi\mathbf{I}_0 + \sin\phi\mathbf{J}_0) + \cos\psi(\cos\theta(-\sin\phi\mathbf{I}_0 + \cos\phi\mathbf{J}_0) + \sin\theta\mathbf{K}_0) \]
\[ \mathbf{K}_3 = -\sin\theta(-\sin\phi\mathbf{I}_0 + \cos\phi\mathbf{J}_0) + \cos\theta\mathbf{K}_0 \]

and the \( \mathbf{k} \) component of the angular momentum is

\[ \mathbf{k} \cdot \mathbf{l} = A(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\sin\psi\sin\theta + B(-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi)\cos\psi\sin\theta + C(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta \]
The only force in the problem is gravity, which acts in the $\mathbf{k}$ direction.

Any gravitational torques will be normal to $\mathbf{k}$, so the angular momentum in the $\mathbf{k}$ direction must be conserved.

\[
p_4 = \left( (A \sin^2 \psi + B \cos^2 \psi) \sin^2 \theta + C \cos^2 \theta \right) \phi + (A - B) \sin \theta \sin \psi \cos \psi \dot{\theta} + C \cos \theta \dot{\psi}
\]

\[
\mathbf{k} \cdot \mathbf{l} = A \left( \dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi \right) \sin \psi \sin \theta + 
B \left( -\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi \right) \cos \psi \sin \theta + C \left( \psi + \phi \cos \theta \right) \cos \theta
\]
Summarize our procedure so far

Write the Lagrangian

Apply holonomic constraints, if any

Assign the generalized coordinates

Find the conjugate momenta

Eliminate conserved variables (cyclic coordinates)

Set up numerical methods to integrate what is left
Let’s take a look at some applications of this method

Flipping a coin

The axisymmetric top

Falling brick
Flipping a coin
A coin is axisymmetric, and this leads to some simplification

\[ L = \frac{1}{2} \left( A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + C(\psi + \phi \cos \theta)^2 \right) + \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz \]

We have \textbf{four} cyclic coordinates, adding \( \psi \) to the set and simplifying \( \phi \)

\[ p_\phi = p_4 = \left( A \sin^2 \theta + C \cos^2 \theta \right) \dot{\phi} + C \cos \theta \dot{\psi} \]

\[ p_\psi = p_6 = C(\dot{\psi} + \dot{\phi} \cos \theta) \]

We can recognize the new conserved term as the angular momentum about \( K \)

\[ I = A(\dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi) I_3 + B(-\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi) J_3 + C(\dot{\psi} + \dot{\phi} \cos \theta) K_3 \]
Flipping a coin

We can use this to simplify the equations of motion

The conserved momenta are constant; solve for the derivatives

\[ \dot{\phi} = \frac{(p_4 - p_6 \cos \theta)}{A \sin^2 \theta}, \quad \dot{\psi} = \frac{p_6}{C} - \frac{(p_4 - p_6 \cos \theta) \cos \theta}{A \sin^2 \theta} \]

(There are numerical issues when \( \theta = 0 \); all is well if you don’t start there)
I can flip it introducing spin about $\mathbf{l}$ with no change in $\phi$ or $\psi$

$$\dot{\phi} = 0 = \dot{\psi}$$

This makes $p_4 = 0 = p_6$

$$p_{\phi} = p_4 = \left( A \sin^2 \theta + C \cos^2 \theta \right) \dot{\phi} + C \cos \theta \dot{\psi} = 0$$

$$p_{\psi} = p_6 = C \left( \dot{\psi} + \dot{\phi} \cos \theta \right) = 0$$

Then

$$\frac{\partial L}{\partial \dot{\theta}} = A \sin \theta \cos \theta \dot{\phi}^2 - C \sin \theta \left( \dot{\psi} + \dot{\phi} \cos \theta \right) = 0$$
Flipping a coin

\[ p_1 = p_{10}, \quad p_2 = p_{20}, \quad p_3 = p_{30} - mgt, \quad p_4 = p_{40} = 0, \quad p_5 = p_{50}, \quad p_6 = p_{60} = 0 \]

from which

\[ q_1 = q_{10} + \frac{p_{10}}{m}, \quad q_2 = q_{20} + \frac{p_{20}}{m}, \quad q_3 = q_{30} + \frac{p_{30}}{m} - \frac{1}{2}gt^2 \]

\[ q_4 = q_{40}, \quad p_5 = \frac{p_{50}}{A}, \quad q_6 = q_{60} \]

or

\[ x = x_0 + ut, \quad y = y_0 + vt, \quad z = z_0 + wt - \frac{1}{2}gt^2 \]

\[ \phi = \phi_0 = 0, \quad \theta = \theta_0 + \omega t, \quad \psi = \psi_0 = 0 \]
Flipping a coin

I claim it would be harder to figure this out using the Euler-Lagrange equations

I’ve gotten the result for a highly nonlinear problem by clever argument augmented by Hamilton’s equations
symmetric top

How about the symmetric top?

Same Lagrangian but constrained

\[
\begin{align*}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= d\mathbf{K} = d \begin{bmatrix} \sin \phi \sin \theta \\ \cos \phi \sin \theta \\ \cos \theta \end{bmatrix}
\end{align*}
\]
symmetric top

Treat the top as a cone

\[ A = \frac{3}{5} m \left( \frac{1}{4} a^2 + h^2 \right) = B, \quad C = \frac{3}{10} ma^2 \]

and apply the holonomic constraints

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = dK = d\begin{bmatrix} \sin \phi \sin \theta \\ \cos \phi \sin \theta \\ \cos \theta \end{bmatrix}
\]
symmetric top

After some algebra

\[
L = \frac{3}{320} m \left( 12a^2 + 31h^2 + (4a^2 - 31h^2) \cos(2\theta) \right) \dot{\phi}^2 + \frac{3}{160} m \left( 4a^2 + 31h^2 \right) \dot{\theta}^2 + \frac{3}{20} ma^2 \dot{\psi}^2
\]

\[
\frac{3}{10} ma^2 \cos \theta \dot{\phi} \dot{\psi} - \frac{3}{4} mgh \cos \theta
\]

and we see that \( \phi \) and \( \psi \) are both cyclic here.

\( \phi \) is the rotation rate about the vertical — the precession

\( \psi \) is the rotation rate about \( K \) — the spin
symmetric top

The conjugate momenta are

\[ p_1 = \frac{3}{160} m \left( (12a^2 + 31h^2) + (12a^2 + 31h^2)\cos(2\theta) \right) \dot{\phi} + \frac{3}{10} ma^2 \cos \theta \dot{\psi} \]

\[ p_2 = \frac{3}{80} m (4a^2 + 31h^2) \dot{\theta} \]

\[ p_3 = \frac{3}{10} ma^2 (\cos \theta \dot{\phi} + \dot{\psi}) \]

There are no external forces and no other constraints, so the first and third are conserved.
We are left with two odes

\[
\dot{\theta} = \frac{80 p_2}{3m(4a^2 + 31h^2)}
\]

\[
\dot{p}_2 = 40 \frac{2(p_1^2 + p_3^2)\cos\theta - p_1 p_3 (3 + \cos(2\theta))}{3m(4a^2 + 31h^2)\sin^3\theta} + \frac{3}{4} mgh \sin\theta
\]

The response depends on the two conserved quantities and these depend on the initial spin and precession rates.

We can go look at this in Mathematica.
Let's go back and look at how the fall of a single brick will go in this method.

I will do the whole thing in Mathematica.
That’s All Folks
Falling brick

\[ p_\phi = p_4 = \left( (A \sin^2 \psi + B \cos^2 \psi) \sin^2 \theta + C \cos^2 \theta \right) \dot{\phi} + (A - B) \sin \theta \sin \psi \cos \psi \dot{\theta} + C \cos \theta \psi \]

What is this?!

If \( A \) and \( B \) are equal (axisymmetric body), this is much simpler

\[ p_\phi = p_4 = (A \sin^2 \theta + C \cos^2 \theta) \dot{\phi} + C \cos \theta \psi \]
Falling brick

We know what happens to all the position coordinates

All that’s left is the single equation for the evolution of $p_5$.

After some algebra we obtain

$$
\dot{p}_5 = -\frac{(p_6 - p_4 \cos \theta)(p_4 - p_6 \cos \theta)}{A \sin^3 \theta}
$$

$$
\dot{\theta} = \frac{p_5}{A}
$$
We can restrict our attention to axisymmetric wheels and we can choose $K$ to be parallel to the axle without loss of generality.

\[
L = \frac{1}{2} \left( A(\dot{\theta} \cos \psi + \dot{\phi} \sin \theta \sin \psi)^2 + B(-\dot{\theta} \sin \psi + \dot{\phi} \sin \theta \cos \psi)^2 + C(\dot{\psi} + \dot{\phi} \cos \theta)^2 \right) \\
+ \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mg \cdot R
\]

\[
L = \frac{1}{2} \left( A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + C(\dot{\psi} + \dot{\phi} \cos \theta)^2 \right) + \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mg \cdot R
\]
If we don’t put in any simple holonomic constraint (which we often can do)

\[
q = \begin{bmatrix}
x \\
y \\
z \\
\phi \\
\theta \\
\psi
\end{bmatrix}
\]
We know \( v \) and \( \omega \) in terms of \( \mathbf{q} \)
any difficulty will arise from \( r \)

\[
v = \omega \times r
\]

\[
r = -aJ_2
\]

Actually, it’s something of a question as to where the difficulties will arise in general

This will depend on the surface

flat, horizontal surface — we’ve been doing this

flat surface — we can do this today

general surface: \( z = f(x, y) \) — this can be done for a rolling sphere
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) + m g_x = \lambda_j C^j_1 \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}^i} \right) + m g_y = \lambda_j C^j_2 \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{z}^i} \right) + m g_z = \lambda_j C^j_3 \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}^i} \right) = \lambda_j C^j_4 \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \lambda_j C^j_5 \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}} \right) = \lambda_j C^j_6
\]
We have the usual Euler-Lagrange equations

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = \lambda_j C^j_i \]

and we can write out the six equations
The angular momentum in body coordinates is

\[
\mathbf{l} = A(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\mathbf{I}_3 + B(-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi)\mathbf{J}_3 + C(\dot{\psi} + \dot{\phi}\cos\theta)I_{zz}\mathbf{K}_3
\]

The body axes are (from Lecture 3)

\[
\mathbf{I}_3 = \cos\psi(\cos\phi\mathbf{I}_0 + \sin\phi\mathbf{J}_0) + \sin\psi(\cos\theta(-\sin\phi\mathbf{I}_0 + \cos\phi\mathbf{J}_0) + \sin\theta\mathbf{K}_0)
\]
\[
\mathbf{J}_3 = -\sin\psi(\cos\phi\mathbf{I}_0 + \sin\phi\mathbf{J}_0) + \cos\psi(\cos\theta(-\sin\phi\mathbf{I}_0 + \cos\phi\mathbf{J}_0) + \sin\theta\mathbf{K}_0)
\]
\[
\mathbf{K}_3 = -\sin\theta(-\sin\phi\mathbf{I}_0 + \cos\phi\mathbf{J}_0) + \cos\theta\mathbf{K}_0
\]

and the \(\mathbf{k}\) component of the angular momentum is

\[
\mathbf{k} \cdot \mathbf{l} = A(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)\sin\psi\sin\theta + \\
B(-\dot{\theta}\sin\psi + \dot{\phi}\sin\theta\cos\psi)\cos\psi\sin\theta + C(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta
\]
We will have a useful generalization fairly soon
The idea of separating the second order equations
into first order equations

\[
\dot{q}^i = \overline{M}^{ir} p_r
\]

\[
\dot{p}_i = \frac{1}{2} \overline{M}^{mr} p_r \frac{\partial M^{mn}}{\partial q^i} \overline{M}^{ns} p_s - \frac{\partial V}{\partial q^i} + \lambda_j C^j_i + Q_i
\]

More generically

\[
\dot{q}^i = A_j^i(q^m) u^j
\]

\[
\dot{u}^i = f^i(q^m, u^n)
\]