5.2 Modal Frequency Response Analysis

Modal frequency response analysis is an alternate approach to computing the frequency response of a structure. This method uses the mode shapes of the structure to reduce the size, uncouple the equations of motion (when modal or no damping is used), and make the numerical solution more efficient. Since the mode shapes are typically computed as part of the characterization of the structure, modal frequency response is a natural extension of a normal modes analysis.

As a first step in the formulation, transform the variables from physical coordinates $[u(\omega)]$ to modal coordinates $[\xi(\omega)]$ by assuming

$$ [x] = [\phi][\xi(\omega)]e^{i\omega t} $$

(5-8)

The mode shapes $[\phi]$ are used to transform the problem in terms of the behavior of the modes as opposed to the behavior of the grid points. Equation (5-8) represents an equality if all modes are used; however, because all modes are rarely used, the equation usually represents an approximation.

To proceed, temporarily ignore all damping, which results in the undamped equation for harmonic motion

$$ -\omega^2[M][\xi(\omega)] + [K][\xi(\omega)] = [P(\omega)] $$

(5-9)

at forcing frequency $\omega$.

Substituting the modal coordinates in Eq. (5-8) for the physical coordinates in Eq. (5-9) and dividing by $e^{i\omega t}$, the following is obtained:

$$ -\omega^2[M][\phi][\xi(\omega)] + [K][\phi][\xi(\omega)] = [P(\omega)] $$

(5-10)

Now this is the equation of motion in terms of the modal coordinates. At this point, however, the equations remain coupled.
To uncouple the equations, premultiply by $[\phi^T]$ to obtain

$$-\omega^2 [\phi^T][M][\phi] [\xi(\omega)] + [\phi^T][K][\phi] [\xi(\omega)] = [\phi^T][P(\omega)]$$

(5-11)

where $[\phi^T][M][\phi] = \text{modal (generalized) mass matrix}$

$[\phi^T][K][\phi] = \text{modal (generalized) stiffness matrix}$

$[\phi^T][P] = \text{modal force vector}$

The final step uses the orthogonality property of the mode shapes to formulate the equation of motion in terms of the generalized mass and stiffness matrices, which are diagonal matrices. These diagonal matrices do not have the off-diagonal terms that couple the equations of motion. Therefore, in this form the modal equations of motion are uncoupled. In this uncoupled form, the equations of motion are written as a set of uncoupled single degree-of-freedom systems as

$$-\omega^2 m_i \xi_i(\omega) + k_i \xi_i(\omega) = p_i(\omega)$$

(5-12)

where $m_i = i$-th modal mass

$k_i = i$-th modal stiffness

$p_i = i$-th modal force

The modal form of the frequency response equation of motion is much faster to solve than the direct method because it is a series of uncoupled single degree-of-freedom systems.

Once the individual modal responses $\xi_i(\omega)$ are computed, physical responses are recovered as the summation of the modal responses using

$$[X] = [\phi][\xi(\omega)] e^{i\omega t}$$

(5-13)

These responses are in complex form (magnitude/phase or real/imaginary) and are used to recover additional output quantities requested in the Case Control Section.
Damping in Modal Frequency Response

If a damping matrix \( [B] \) exists, the orthogonality property (see Section 3.2) of the modes does not, in general, diagonalize the generalized damping matrix

\[
[\phi]^T [B] [\phi] \approx \text{diagonal}
\]  
(5-14)

If structural damping is used, the orthogonality property does not, in general, diagonalize the generalized stiffness matrix

\[
[\phi]^T [K] [\phi] \approx \text{diagonal}
\]  
(5-15)

where \( [K] = (1 + iG) [K] + i \sum G_E [K_E] \)

In the presence of a \( [B] \) matrix or a complex stiffness matrix, the modal frequency approach solves the coupled problem in terms of modal coordinates using the direct frequency approach described in Section 5.1

\[
\begin{bmatrix}
\end{bmatrix}
[\xi(\omega)] = [\phi]^T [P(\omega)]
\]  
(5-16)

Equation (5-16) is similar to Eq. (5-5) for the direct frequency response analysis method except that Eq. (5-16) is expressed in terms of modal coordinates \( \xi \). Since the number of modes used in a solution is typically much less than the number of physical variables, using the coupled solution of the modal equations is less costly than using physical variables.

If damping is applied to each mode separately, the uncoupled equations of motion can be maintained. When modal damping is used, each mode has damping \( b_i \) where \( b_i = 2m_i \omega_i \xi_r \). The equations of motion remain uncoupled and have the form

\[
-\omega^2 m_i \xi_i(\omega) + i \omega b_i \xi_i(\omega) + k_i \xi_i(\omega) = p_i(\omega)
\]  
(5-17)

for each mode.

Each of the modal responses is computed using

\[
\xi_i(\omega) = \frac{p_i(\omega)}{-m_i \omega^2 + i b_i \omega + k_i}
\]  
(5-18)
The TABDMP1 Bulk Data entry defines the modal damping ratios. A table is created by the frequency/damping pairs specified on the TABDMP1 entry. The solution refers to this table for the damping value to be applied at a particular frequency. The TABDMP1 Bulk Data entry has a Table ID. A particular TABDMP1 table is activated by selecting the Table ID with the SDAMPING Case Control command.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>TABDMP1</td>
<td>TID</td>
<td>TYPE</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>g1</td>
<td>f2</td>
<td>g2</td>
<td>f3</td>
<td>g3</td>
<td>etc.-</td>
<td>ENDT</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Field**

**Contents**

TID: Table identification number.

TYPE: Type of damping units:

- G (default)
- CRIT
- Q

fi: Frequency value (cycles per unit time).

gi: Damping value in the units specified.

At resonance, the three types of damping are related by the following equations:

\[
\xi_i = \frac{b_i}{b_{cr}} = \frac{G_i}{2}
\]

\[
b_{cr} = 2m_i \omega_i
\]

\[
Q_i = \frac{1}{(2\xi_i)} = \frac{1}{G_i}
\]

Note that the \(i\) subscript is for the \(i\)-th mode, and not the \(i\)-th excitation frequency.

The values of \(fi\) and \(gi\) define pairs of frequencies and dampings. Note that \(gi\) can be entered as one of the following: structural damping (default), critical damping, or quality factor. The entered damping is converted to structural damping internally using Eq. (5-19). Straight-line interpolation is used for modal frequencies between consecutive \(fi\) values. Linear extrapolation is used at the ends of the table. ENDT ends the table input.

For example, if modal damping is entered using Table 5-1 and modes exist at 1.0, 2.5, 3.6, and 5.5 Hz, MSC/NASTRAN interpolates and extrapolates as shown in Figure 5-3 and in the table. Note that there is no table entry at 1.0 Hz; MSC/NASTRAN uses the first two table entries at \(f = 2.0\) and \(f = 3.0\) to extrapolate the value for \(f = 1.0\).
Modal damping is processed as a complex stiffness when PARAM,KDAMP is entered as -1. The uncoupled equation of motion becomes

\[-\omega^2 m_j \xi_j(\omega) + (1 + iG(\omega)) k_j \xi_j(\omega) = p_i(\omega)\]  \hspace{1cm} (5-20)

The default for PARAM,KDAMP is 1, which processes modal damping as a damping matrix as shown in Eq. (5-17).
The decoupled solution procedure used in modal frequency response can be used only if either no damping is present or modal damping alone (via TABDMP1) is used. Otherwise, the modal method uses the coupled solution method on the smaller modal coordinate matrices if nonmodal damping (i.e., CVISC, CDAMPi, GE on the MATi entry, or PARAM,G) is present.

Mode Truncation in Modal Frequency Response Analysis

It is possible that not all of the computed modes are required in the frequency response solution. You need to retain, at a minimum, all the modes whose resonant frequencies lie within the range of forcing frequencies. For example, if the frequency response analysis must be between 200 and 2000 Hz, all modes whose resonant frequencies are in this range should be retained. This guideline is only a minimum requirement, however. For better accuracy, all modes up to at least two to three times the highest forcing frequency should be retained. In the example where a structure is excited to between 200 and 2000 Hz, all modes from 0 to at least 4000 Hz should be retained.

The frequency range selected on the eigenvalue entry (EIGRL or EIGR) is one means to control the modes used in the modal frequency response solution. Also, three parameters are available to limit the number of modes included in the solution. PARAM,LFREQ gives the lower limit on the frequency range of retained modes, and PARAM,HFREQ gives the upper limit on the frequency range of retained modes. PARAM,LMODES gives the number of lowest modes to be retained. These parameters can be used to include the proper set of modes. Note that the default is for all computed modes to be retained.

Dynamic Data Recovery in Modal Frequency Response Analysis

In modal frequency response analysis, two options are available for recovering displacements and stresses: the mode displacement method and the matrix method. Both methods give the same answers, although with differences in cost.

The mode displacement method computes the total physical displacements for each excitation frequency from the modal displacements, and then computes element stresses from the total physical displacements. The number of operations is proportional to the number of excitation frequencies.

The matrix method computes displacements per mode and element stresses per mode, and then computes physical displacements and element stresses as the summation of modal displacements and element stresses. Costly operations are proportional to the number of modes.

Since the number of modes is usually much less that the number of excitation frequencies, the matrix method is usually more efficient and is the default. The mode displacement method can
The mode displacement method is selected by using PARAM,DDRMM,-1 in the Bulk Data. The mode displacement method is used when "frequency-frozen" structural plots are requested (see Chapter 9).

The mode acceleration method (Chapter 11) is another data recovery method for modal frequency response analysis. This method can provide better accuracy since detailed local stresses and forces are subject to mode truncation and may not be as accurate as the results computed with the direct method.
**MODAL FREQUENCY RESPONSE** = SOL 111

Solves same SS sine response as above, but works in modal space.

First solve for natural frequencies (undamped)
\[ \begin{bmatrix} K - \omega_n^2 M \end{bmatrix} \{ \Phi \} = \{ 0 \} \quad \text{with } \omega_n \text{ and } \Phi \text{ output} \]
\[ \Phi = \text{eigenvector} = \text{vibrational mode shape (arbitrarily scaled)} \]
\[ \omega_n = \text{natural frequency (radians)} = 2\pi f \quad (f = \text{Hz} = \text{cycles/sec}) \]
Best all-purpose technique = Lanczos method => use **eigrl**

Substitute \( U = \Phi z \) where \( z = \text{modal participation coefficient} \)

Substitute into dynamics eqn and premultiply by \( \Phi^T \)
\[ \Phi^T \left[ -\omega^2 M + i\omega B + K \right] \Phi z = \Phi^T P \]

Thus \[ \left[ -\omega^2 m + i\omega b + k \right] z = p \]
where
\[ m = \Phi^T M \Phi = \text{generalized mass} \]
\[ k = \Phi^T K \Phi = \text{generalized stiffness} = -\omega^2 m \]
\[ b = \Phi^T B \Phi = \text{generalized damping} = \omega g(\omega) m \]
\[ \text{where } g(\omega) \text{ specified on TABDMP} \]
\[ p = \Phi^T P = \text{generalized force} \]

Solve the above uncoupled (1 DOF) equations (1 for each natural freq)

Convert back to physical space \( U = \Phi z \)

**MODAL FREQUENCY RESPONSE** = SOL 111

* User specifies model => K,M
* User specifies eigenvalue technique = **EIGRL**
* User specifies amplitude of load = \( P \) on **RLOAD,DAREA,TABLED1**
* User specifies freq list at which want solutions = \( f \) on **FREQ** card
* User specifies damping by **SDAMP,TABDMP1**
* NASTRAN solves for amplitude of response = \( U,\dot{U},\ddot{U},\sigma, \text{etc} \)
* Note: \( e^{i\omega t} \) is implied, and not identified in load or response
* Output requests include **DISP,VELO,ACCE,STRESS**
* Output modes by **VECTOR**, output z by **SDISP,SVELO,SACCE**
* **SORT1** for all response at 1 freq, **SORT2** for 1 response at all freq
* To send response to XL, use **param,post,0** can plot contours or XY
MODAL FREQUENCY RESPONSE ANALYSIS
STEADY STATE HARMONIC RESPONSE
MODAL APP = FIND NAT FREQ FIRST

TIME 5 (DYNAMICS REQUIRE LARGER TIME LIMITS)
SOL 111 (MODAL FREQUENCY ANALYSIS)
CEND
TITLE=
SPC=9 (ONLY 1 BC SET PER RUN)
MPC=11 (TURN ON MPC EQUATIONS)
METHOD=13 (CALL EIGRL)
SVECTOR= (PRINT MODE SHAPES)
SDAMP=19 (CALL TABDMP = DAMPING)
SUBCASE 1 (1 SUBCASE FOR EACH LOAD CONDITION)
LABEL=
DLOAD=20 (DYNAMIC LOAD: CALL RLOADi or DLOAD)
FREQ=30 (FREQUENCIES SOLVED FOR: CALL FREQ)
DISP(SORT2,PHASE)= (OUTPUT REQUEST FOR DISP AMPLITUDE)
ACCE(SORT2,PHASE)= (OUTPUT REQUEST FOR ACCE AMPLITUDE)
STRESS(SORT2,PHASE)= (OUTPUT REQUEST FOR STRESS AMPLITUDE)
SUBCASE 2 (ADDITIONAL LOAD CASES)
...
BEGIN BULK
GRIDS,ELEMENTS,PROPERTIES... (MODEL DESCRIPTION)
SPC,9... (BC FOR THIS RUN - ONLY 1 SET/RUN ACTIVE)
MPC,11... (MPC IF DESIRED)
MAT1... (INCLUDE MASS DENSITY FOR ALL ELEMENTS)
CONN2... (LUMPED MASSES)
PARAM,GRDPNT,0 (PRINT MASS PROPERTY TABLE)
PARAM,POST,0 (SEND OUTPUT TO XL)
...
EIGRL,13... (DEFINE EIGENVALUE TECHNIQUE)
...
DLOAD,20,1,1,1,21 (COMBINES RLOADi CARDS)
RLOAD2,21,22,23,24 (DEFINE LOAD AMPLITUDE: CALL DAREA,TABLED)
DAREA,22 (POINT OF LOAD APLICATION)
TABLEDi,24... (LOAD AMPLITUDE AS FUNCTION OF FREQUENCY)
FREQ1,30... (LIST FREQUENCIES (f) AT WHICH WANT SOLUTION)
...
TABDMP1,19... (DEFINE DAMPING AS FUNCTION OF FREQ)
ENDDATA

REF: MSC/Handbook for Dynamic Analysis = HDA
For complete list of damping options HDA Sec 3.2

Note: In this solution, you describe load amplitude and phase (f)
Results are response amplitude and phase (u). The sin(wt) implied.
$ 2 Runs: NATURAL FREQUENCY FOLLOWED BY HARMONIC RESPONSE
$ 1st Run: calc natural freq, save database
$
SOL 103 (NATURAL FREQUENCY ANALYSIS)
$
... 
CEND 
SPC=10 
MPC=20 
METHOD=13 
...
BEGIN BULK 
..GRIDS/ELEMENTS/BC 
..PROPERTIES W MASS DENSITY 
EIGRL,13... 
ENDDATA 
$>
nastran run1 scr=no (creates run1.dball = database)
$
RESTART (RESTART FROM EXISTING DATABASE)
ASSIGN MASTER='run1.master' (POINTS TO run1 DATABASE)
SOL 111 (MODAL FREQUENCY ANALYSIS)
...
CEND 
TITLE= 
SPC=10 (MUST KEEP SAME SPC AS RUN1) 
MPC=20 (MUST KEEP SAME MPC AS RUN1) 
METHOD=13 (MUST KEEP SAME METHOD AS RUN1) 
DLOAD=20 (DYNAMIC LOAD: CALL RLOADi or DLOAD) 
FREQ=30 (FREQUENCIES SOLVED FOR: CALL FREQ) 
SDAMP=19 (CALL TABDMP) 
DISP(SORT2,PHASE)= (OUTPUT REQUEST FOR DISP AMPLITUDE) 
STRESS(SORT2,PHASE)= (OUTPUT REQUEST FOR STRESS AMPLITUDE) 
...
BEGIN BULK 
***Only bulk data is LOAD input, do NOT include model=GRIDS,element***** 
...
DLOAD,20,1,1,21 (USE LMODES OR LFREQ & HFREQ - NOT BOTH) 
RLOAD2,21,22,23,24 (DEFINE LOAD AMPLITUDE: CALL DAREA,DPHASE,TABLED) 
DAREA,22 (POINT OF LOAD APPLICATION) 
DPHASE,23... (PHASE ANGLE OF LOAD) 
TABLDi,24... (LOAD AMPLITUDE AS FUNCTION OF FREQUENCY) 
FREQ1,30... (LIST FREQUENCIES (f) AT WHICH WANT SOLUTION) 
TABDMP1,19... (DEFINE DAMPING AS FUNCTION OF FREQ) 
ENDDATA 
$>
nastran run2 (FORCED RESPONSE BASED ON RUN1 MODES)
$

Note: These databases can get quite large (disk storage)
Must keep both run1.dball and run1.master - req'd for restarts.
You can delete run1.usrsou and run1.usrobj - not needed.
MODAL FREQUENCY RESPONSE - 2 DOF SPRING/MASS SYSTEM
Ref: Thomson, Vibrations 3rd Ed, Example 5.3-1

Note: UNITS MUST BE CONSISTENT
Solution 111 is steady-state harmonic response
via "modal" approach, NOT direct

\( m_{\text{freq.dat}} \)

time 5
sol 111
$modal freq resp
cend
title=2 dof spring mass system
subtitle=Thomson example 5.3-1
spc=70
$turn on bc
method=90
$call eigrl
svector=all
$print mode shapes
subcase 1
label=ss sine force on grid 51
dload=80
$call rload2 (dyn load)
freq=85
$call freq (f steps)
sdamp=44
$call tabdmp (damping)
set 8=51,52
disp(sort2,real)=8
$real=>real/imag
$disp(sort2,phase)=8
$phase=>magnitude/phase
sdisp(sort2,real)=all
$print modal participation coeff
begin bulk
param,post,0
$send output to XL
param,grdpnt,0
$print mass prop

$GRID POINTS WITH 5 DOF FIXED
grid,50,,0,,0,,0,,0,,23456
grid,51,,1,,0,,0,,0,,23456
grid,52,,2,,0,,0,,0,,23456
grid,53,,3,,0,,0,,0,,23456

$SPRINGs CONNECTING X DISPL
celas1,21,20,50,1,51,1
celas1,22,20,51,1,52,1
celas1,23,20,52,1,53,1
pelas,20,18
\( K=18 \)

$LUMPED MASSES
conm2,31,51,,2
\( M=2 \)
conm2,32,52,,2

$FIXED BC AT WALLS
spcl,70,1,50,53

$APPLIED LOAD
rload2,80,81,,2

darea,81,51,1,18
$force on grid
tabled,82
$force vs freq
,0,,1,,1000,,1,,endt
freql,85,0,,0,0,,5,30
$freq=Hz, not radians/sec
\( \% \) of critical damping added to prevent infinite
alternatives = cvisc,cдamp,g

tabdmp1,44,cr
damping vs freq
,0,,0,0,,1,,100,,0,1
endt

$EIGENVALUE CALCULATION
eigrl,90,0,,2
ENDDATA

\( \text{Figure 5.3-2. Forced response of the two-DOF system.} \)
DAMPING INPUT (Ref H/DA Sec 3.2)

VISCOS DAMPING (Proportional to velocity = B ü)
*input on CVISC or CDAMP
*used for air damping/shock absorbers
*available in Direct Solutions
*for Modal Solutions see Modal Damping below

STRUCTURAL DAMPING (Proportional to displacement = i g K)
*input g on MAT card (per material) or PARAM.C (total K)
*used for internal material friction and yield, and for damping due to bolted/riveted connections
*commonly used values:
  welded steel  g = 0.04 - 0.08
  bolted steel  g = 0.06 - 0.08
  concrete  g = 0.08 - 0.04
*available in Direct Solutions
*for Modal Solutions see Modal Damping below

MODAL DAMPING (User defined = b ë)
*input g(f) on TABDMP bulk data called by SDAMP case control
*can define any form of damping thru g(f) where:
  b = g(ω) k / w = g(ω) m w
*can also use "fraction of critical = C/Cₖ" or "amplification quality factor = Q"

SELECTION OF SOLUTION TECHNIQUE

MODAL is usually best:
*most often used in practice
*efficient when few modes sufficient to predict response
*natural problem flow
  -model checkout
  -static solution (SOL 101)
  -natural frequency solution (SOL 103)
  -forced response (SOL 11/12)
*if checkpoint SOL 103 then SOL 11 efficient
*care must be used with this technique

DIRECT is good when:
*models have small bandwidth
*large # of modes required in modal method
*structure is highly damped