INTRODUCTION TO INERTIAL CONFINEMENT FUSION

R. Betti

Lecture 6

Ignition of a simple DT plasma
Recap from previous lecture

- Quasi-static approximation of a stagnating plasma where the alpha heating starts at stagnation leads to an ignition condition known as the Lawson ignition criterion

\[ \chi_{no\alpha} \equiv \frac{P_{no\alpha}}{24 / S\varepsilon_\alpha} > 1 \quad S \equiv \frac{\langle \sigma v \rangle}{T^2} \text{ at about } 8\text{keV} \]

- Minimum \( P\tau \) for ignition is

\[ \left[ P_{no\alpha} \tau \right]_{\text{ign}} \equiv \frac{24}{S\varepsilon_\alpha} \]

\[ \left[ P_{no\alpha} \tau \right]_{\text{ign}} \equiv 1.1\times10^6 \text{ Pa}\cdot\text{s} \approx 11\text{ atm}\cdot\text{s} \approx 11\text{ Gbar}\cdot\text{ns} \]

- This is called Lawson triple product \( P\tau = 2nT\tau \)

- A better ignition model uses a dynamic evolution of a thin and dense cold shell/piston compressing a hot “hot spot” plasma
Hot spot ignition: dynamic model
The isobaric approximation for the hot spot
From previous lecture

- The shell has been accelerated (by the laser) to a peak implosion velocity.
- The shell is compressing the hot spot and raising its pressure by converting its kinetic energy into hot spot internal energy.
- The hot spot is gaining energy from the shell and from alpha heating while losing energy through radiation emission and losing heat through heat conduction.

- The simplest shell model is the one of a thin dense shell (high aspect ratio) whose motion (after being accelerated by the laser) is governed by Newton’s law.

\[ M_{sh} \ddot{R} = 4\pi PR^2 \]

\[ R(0) = R_* \]

\[ \dot{R}(0) = -V_i \]

- P(t) is the hot spot pressure pushing on the shell surface $4\pi R^2$
- t=0 is now the time of peak implosion velocity or beginning of the coasting/deceleration phase
Need to couple Newton’s law for the shell to the energy equation and momentum equation of the hot spot because there are two unknowns: \( P(r,t) \) and \( R(t) \). Note that in general the pressure inside the hot spot is a function of radius and time.

Start with the hot spot momentum equation (Euler equation):

\[
\rho \left( \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial r} \right) = -\frac{\partial P}{\partial r}
\]

Scale each term. Here \( \leftrightarrow \) means “compare”

\[
\rho \left( \frac{U}{t} + U \frac{U}{r} \right) \leftrightarrow -\frac{P}{r}
\]

Time scale: \( t \sim R/V_i \); velocity: \( U \sim V_i \); Spatial scale \( r \sim R \)

\[
\rho \left( \frac{V_i^2}{R} + \frac{V_i^2}{R} \right) \leftrightarrow \frac{P}{R}
\]
From previous lecture

- Define the Mach number of the hot spot

\[
Mach^2 = \frac{V_i^2}{P / \rho} \sim \frac{V_i^2}{T / m_i} \sim \frac{V_i}{\nu_{th}^2}
\]

- Therefore the LHS is of order \(Mach^2\) with respect to the RHS

\[
\rho \left( \frac{V_i^2}{R} \right) \sim Mach^2 \frac{P}{R}
\]

- Hot spot is multi keV and the thermal velocity of its plasma is >> implosion velocity of the shell, therefore \(Mach<1\): **SUBSONIC HOT SPOT**

\[
Mach^2 \sim \frac{V_i^2}{\nu_{th}^2} \ll 1
\]

- Therefore neglect LHS of momentum equation leads to \(P=P(t)\): **ISOBARIC HOT SPOT**

\[
\frac{\partial P}{\partial r} \approx 0 \quad P \approx P(t)
\]
The dynamic energy balance for the hot spot
Start with the exact specific energy equation (energy per unit volume) for an ideal plasma

\[ e_{HS} = \frac{3}{2}P + \frac{1}{2} \rho U^2 \]  \( \leftarrow \) Total specific energy: internal + kinetic

\[ \frac{\partial e_{HS}}{\partial t} = \nabla \cdot \left[ -\vec{U} \left( e_{HS} + P \right) + \kappa \nabla T \right] + \dot{q}_\alpha - \dot{q}_{rad} \]

- Rate of change
- Flux of enthalpy
- Heat flux
- Alpha heating
- Radiation losses

\[ h_{HS} \equiv e_{HS} + P = \frac{5}{2}P + \frac{1}{2} \rho U^2 \]  \( \leftarrow \) Enthalpy for unit volume

Note the pdV work is not fully into this equation. Since the pdV work acts on the surface of the plasma, its contribution comes when the above equation is integrated over the hot spot volume.
• Using the results of previous section: isobaric approximation and subsonic hot spot flow → leads to

\[ e_{HS} = \frac{3}{2} P + \frac{1}{2} \rho U^2 \approx \frac{3}{2} P(t) \left[ 1 + \frac{U^2}{3P/\rho} \right] \approx \frac{3}{2} P(t) \left[ 1 + O(Mach^2) \right] \]

Neglect for Mach<<1

\[ h_{HS} = \frac{5}{2} P + \frac{1}{2} \rho U^2 \approx \frac{5}{2} P(t) \left[ 1 + O(Mach^2) \right] \]

• Use the bremmstrahlung formula for DT (Z=1) derived in Lecture 3

\[ \dot{q}_{rad} = C_b n^2 \sqrt{T} \]

• Combine with alpha heating

\[ \dot{q}_\alpha - \dot{q}_{rad} = n^2 \left[ \frac{\varepsilon_\alpha}{4} \left\langle \sigma v \right\rangle - C_b \sqrt{T} \right] = \frac{P^2}{4} \left[ \frac{\varepsilon_\alpha}{4} \frac{\left\langle \sigma v \right\rangle}{T^2} - \frac{C_b}{T^{3/2}} \right] \]
\[
\dot{q}_\alpha - \dot{q}_{\text{rad}} = P^2 \Pi(T) \\
\Pi(T) \equiv \frac{1}{4} \left[ \frac{\varepsilon_\alpha}{4} \frac{\langle \sigma v \rangle}{T^2} - \frac{C_b}{T^{3/2}} \right]
\]

- Simplified energy equation for subsonic/isobaric hot spots. Valid in 3D

\[
\frac{3}{2} \frac{dP(t)}{dt} \approx \nabla \cdot \left[ -\vec{U} \left( \frac{5}{2} P(t) \right) + \kappa \nabla T \right] + P(t)^2 \Pi(T)
\]

- Integrate over volume \( V \) enclosed by a surface \( S \) and apply Gauss theorem

\[
\frac{3}{2} V \frac{dP(t)}{dt} \approx \int_S \left[ -\vec{U} \vec{n} \left( \frac{5}{2} P(t) \right) + \kappa \vec{n} \cdot \nabla T \right] dS + P(t)^2 \int_V \Pi(T) dV
\]
Consider following simple model of hot spot hydrodynamic profiles:

- Note while the flat pressure profile was derived from the subsonic flow approximation, the flat profiles of $T$ and $\rho$ are not derived. For now, they are simply assumed. IMPORTANT: the product of the profiles $T \times \rho$ must be flat to satisfy the ideal gas EOS inside the hot spot.

\[
P(t) = 2nT = \frac{2}{m_i} \rho T
\]
The continuity of the pressure across the hot spot-shell interface implies that the shell temperature must be a lot less than the hot spot temperature (cold shell)

\[ \rho_{HS} T_{HS} \approx \rho_{Sh} T_{Sh} \]

\[ T_{Sh} \approx \frac{\rho_{HS}}{\rho_{Sh}} T_{HS} \ll T_{HS} \]

Since the thermal conductivity of a plasma is proportional to \( T^{5/2} \), the shell behaves like a thermal insulator when compared to the hot spot

\[ \kappa_{Spitzer}(T_{Sh}) \approx \left( \frac{T_{Sh}}{T_{HS}} \right)^{5/2} \]

\[ \kappa_{Spitzer}(T_{HS}) \approx \left( \frac{\rho_{HS}}{\rho_{sh}} \right)^{5/2} \]

\[ \kappa_{Spitzer}(T_{HS}) \ll \kappa_{Spitzer}(T_{HS}) \]

This implies we can neglect the heat flux inside the shell
• Consider mass conservation equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = 0 \]

• Integrate the equation through the volume \( \Delta V \) across the hot spot-shell interface. Note \( \Delta V \) can be any volume element across that interface

\[ \int_{\Delta V} \frac{\partial \rho}{\partial t} dV + \int_{\Delta S^+} \left[ \rho \vec{U} \cdot \vec{n} \right] dS + \int_{\Delta S^-} \left[ \rho \vec{U} \cdot \vec{n} \right] dS = 0 \]

• Bring volume integral inside time derivative

\[ \frac{\partial}{\partial t} \left( \int_{\Delta V} \rho dV \right) - \int_{\Delta S^-} \left[ \rho \vec{U}_V \cdot \vec{n} \right] dS + \int_{\Delta S^+} \left[ \rho \vec{U} \cdot \vec{n} \right] dS = 0 \]

• Combine integrals

\[ \frac{\partial}{\partial t} \left( \int_{\Delta V} \rho dV \right) + \int_{\Delta S^-} \left[ \rho \left( \vec{U} - \vec{U}_V \right) \cdot \vec{n} \right] dS = 0 \]
• Take limit of $\Delta V \to 0$

$$\int_{\Delta S^{-}}^{\Delta S^{+}} [\rho \left( \vec{U} - \vec{U}_V \right) \cdot \vec{n}] dS = 0$$

• Valid for arbitrary $\Delta S$, therefore the equality below applies to every point on the surface and implies that the mass flow across the interface is continuous (NOT SURPRISING!)

$$\left[ \rho \left( \vec{U} - \vec{U}_V \right) \cdot \vec{n} \right]_{S^{-}} = \left[ \rho \left( \vec{U} - \vec{U}_V \right) \cdot \vec{n} \right]_{S^{+}}$$

• Since $S^{-}$ is inside the hot spot and $S^{+}$ is inside the shell then and $\vec{n}$ is radial in 1D

$$\rho_{HS} V_b = \rho_{Sh} V_a \quad V_a \equiv \left| U - U_V \right|_{Sh} \quad V_b \equiv \left| U - U_V \right|_{HS}$$

$$V_a = \frac{\rho_{HS}}{\rho_{Sh}} V_b \ll V_b \quad \rho_{HS} \ll \rho_{Sh}$$

• Note that $V_a$ and $V_b$ are flow velocities relative to the interface moving with $U_V$. 
• Continuity of the mass flow across the interface and the fact that the shell density is much greater than the hot spot density leads to a relative flow much faster in the hot spot than in the shell (If the flow exists) Here relative means relative to the interface.

\[ \rho_{HS} V_b = \rho_{Sh} V_a \]

Here \( \rho_{HS} \) is the density of the hot spot and \( \rho_{Sh} \) is the density of the shell. The relative flow is given by:

\[ V_a = \frac{\rho_{HS}}{\rho_{Sh}} V_b \ll V_b \]

\[ \rho_{HS} \ll \rho_{Sh} \]

Note that \( V_a \) and \( V_b \) are flow velocities relative to the interface moving with \( U_v \).

• But does mass exchange really occur between shell and hot spot or both \( V_a \) and \( V_b \) are zero? More later
• Consider the integration of the energy equation up to a surface $S^+$ and radius $R^+$ just inside the shell so we can neglect the heat flux

$$\frac{3}{2} V \frac{dP(t)}{dt} \approx \int_{S^+} \left[ -\vec{U} \cdot \vec{n} \left( \frac{5}{2} P(t) \right) + \kappa \vec{n} \cdot \nabla T \right] dS + P(t)^2 \int_V \Pi(T) dV$$

• Valid in 3D

$$\frac{3}{2} V^+ \frac{dP(t)}{dt} \approx \left( \frac{5}{2} P(t) \right) \int_{S^+} \left[ -\vec{U} \cdot \vec{n} \right] dS + P(t)^2 \int_{V^+} \Pi(T) dV$$

• Use the general volume evolution equation. $U_v$ is the velocity at which the volume is changing

$$\frac{dV^+}{dt} = \int_{S^+} \left[ \vec{U}_v \cdot \vec{n} \right] dS$$

• Rewrite:

$$\int_{S^+} \left[ \vec{U} \cdot \vec{n} \right] dS = \int_{S^+} \left[ \left( \vec{U} - \vec{U}_v \right) \cdot \vec{n} \right] dS + \frac{dV^+}{dt}$$
• Rewrite energy equation:

\[
\frac{3}{2} V^+ \frac{dP(t)}{dt} \approx -\left(\frac{5}{2} P(t)\right) \frac{dV^+}{dt} + P(t)^2 \int_{V^+} \Pi(T) dV + \left(\frac{5}{2} P(t)\right) \int_{S^+} \left[ (\vec{U} - \vec{U}_v) \cdot \vec{n} \right] dS
\]

• Combine terms (and drop superscript + from V but not from S):

\[
\frac{3}{2} \frac{1}{V^{2/3}} \frac{d}{dt} \left( PV^{5/3} \right) \approx P(t)^2 \int_{V} \Pi(T) dV + \left(\frac{5}{2} P(t)\right) \int_{S^+} \left[ (\vec{U} - \vec{U}_v) \cdot \vec{n} \right] dS
\]

This term represents the flow $V_a$ across the surface $S^+$ inside the shell just outside the hot spot.

\[
\frac{3}{2} \frac{1}{V^{2/3}} \frac{d}{dt} \left( PV^{5/3} \right) \approx P(t)^2 \int_{V} \Pi(T) dV + \left(\frac{5}{2} P(t)\right) \int_{S^+} [V_a] dS
\]
• From two previous slides: \[ V_a = \frac{\rho_{HS}}{\rho_{Sh}} V_b \]

• Substitute into energy equation and take limit for \( \rho_{HS} \ll \rho_{Sh} \)

\[
\frac{3}{2} \frac{1}{V^{2/3}} \frac{d}{dt} \left( PV^{5/3} \right) \approx P(t)^2 \int V \Pi(T) dV + \left( \frac{5}{2} P(t) \right) \int_{S^+} \frac{\rho_{HS}}{\rho_{Sh}} V_b dS
\]

neglect

• Energy equation for isobaric hot spot surrounded by a shell with “infinite” density

\[
\frac{3}{2} \frac{1}{V^{2/3}} \frac{d}{dt} \left( PV^{5/3} \right) \approx P(t)^2 \int V \Pi(T) dV
\]
Ablation off the inner shell surface
To estimate the flow across $S^+$, let’s take the integral of the energy equation over $\Delta V$ across interface i.e. from just inside to just outside the hot spot

$$\frac{3}{2} \frac{dP(t)}{dt} \Delta V \approx \int_{\Delta S^-}^{\Delta S^+} \left[ -\vec{U} \cdot \vec{n} \left( \frac{5}{2} P(t) \right) + \kappa \vec{n} \cdot \nabla T \right] dS + P(t) \Pi(T) \Delta V$$

In the limit of $\Delta V \rightarrow 0$ we find (this limit is taken by setting $\Delta V = \Delta S dr$ and taking $dr \rightarrow 0$)

$$\left[ -\vec{U} \cdot \vec{n} \left( \frac{5}{2} P(t) \right) + \kappa \vec{n} \cdot \nabla T \right]_{\Delta S^+} = \left[ -\vec{U} \cdot \vec{n} \left( \frac{5}{2} P(t) \right) + \kappa \vec{n} \cdot \nabla T \right]_{\Delta S^-}$$

Neglect this because inside the shell where $\kappa \rightarrow 0$

For arbitrary $\Delta S$, heat losses from hot spot compensated by enthalpy flux

$$\left( \vec{U}^- - \vec{U}^+ \right) \cdot \vec{n} \left( \frac{5}{2} P(t) \right) = \left[ \kappa \vec{n} \cdot \nabla T \right]_{S^-}$$

Heat flux leaving the hot spot $<0$ because $\nabla T$ negative at $R^-$. Therefore flow is inward
• Replace from before (relative flow is inward, i.e. negative $e_r$)

$$\left(\vec{U}^- - \vec{U}^+\right) \cdot \hat{n} = - (V_b - V_a)$$

• Rewrite previous equation. Heat losses are compensated by enthalpy flux across the interface

$$-(V_b - V_a) \left( \frac{5}{2} P(t) \right) = \left[ \kappa \hat{n} \cdot \nabla T \right]_{S-}$$

• Since

$$V_a = \frac{\rho_{HS}}{\rho_{Sh}} V_b \ll V_b$$

$$V_b \approx - \frac{2}{5} \left[ \kappa \hat{n} \cdot \nabla T \right]_{S-}$$

$$V_a \approx - \frac{2}{5} \frac{\rho_{HS}}{\rho_{Sh}} \left[ \kappa \hat{n} \cdot \nabla T \right]_{S-}$$
- $V_a$ is the penetration velocity of the hot spot boundary into the shell, also called **ABLATION VELOCITY**

![Diagram showing flow in the frame of reference of the shell]
- $V_b$ is the velocity at which the ablated shell plasma blows off into the hot spot, also called the **BLOW-OFF VELOCITY**

- Ablation (and blow off) of the shell into the hot spot is driven by the heat flux from the hot spot
- Heat flows out of the hot spot following the temperature gradient (high $T$ inside hot spot and low $T$ inside the shell)

**Diagram:**
- **Hotspot-shell interface**
- **Frame of Reference.**

**Equations:**
- Enthalpy flux into hot spot: $\frac{5}{2} P V_b$
- Heat flux out of hot spot: $-\kappa \frac{dT}{dr}$
- No heat goes through shell (cold shell = thermal insulator)
Locating hotspot-shell interface at $R^+$ inside cold shell where there is not heat flux makes the hot spot ADIABATIC (in the absence of alpha heating and radiation losses)

ADIABATIC here means there is no heat flux across the boundary

$$\frac{3}{2} V^{2/3} \frac{d}{dt} \left( PV^{5/3} \right) \approx P(t)^2 \int_V \Pi(T) dV$$

- If RHS = 0 (no alpha heating and no radiation losses) then

$$\frac{d}{dt} \left( PV^{5/3} \right) \approx 0 \quad PV^{5/3} \approx const$$
• Important difference: an Adiabatic hot spot does NOT mean that the plasma inside the hot spot has constant Entropy

\[ Adiabatic \neq isentropic \]

• Entropy of a plasma (or gas) is a local quantity related to a fluid element. Below is the entropy per unit mass. The constant \( c_p \) is the specific heat.

\[
s = c_p \log \frac{P}{\rho^{5/3}}
\]

• In a closed system, mass \( M \) is conserved and

\[
\rho = \frac{M}{V} \quad \frac{P}{\rho^{5/3}} = \frac{PV^{5/3}}{M^{5/3}}
\]

Therefore if \( M \) is constant and \( PV^{5/3} \) is constant then Entropy is constant. This however does NOT apply to the hot spot which is an open system receiving mass from the shell through ablation while losing heat. In the hot spot \( M \) is not constant
High temperature limit and the simplest dynamic model
It is safe to state that if the hot spot is heated by compression to temperatures of about 6keV or higher then the alpha heating greatly exceeds the radiation losses and the rad losses can be neglected.
• From before

\[ \dot{q}_\alpha - \dot{q}_{\text{rad}} = P^2 \Pi(T) \]

\[ \Pi(T) \equiv \frac{1}{4} \left[ \frac{\varepsilon_\alpha}{4} \frac{\langle \sigma v \rangle}{T^2} - \frac{C_b}{T^{3/2}} \right] \]

Neglect rad losses

• If we consider temperatures in the 8 to 23 keV range, then \( \langle \sigma v \rangle \approx S_f T^2 \)
• In this high temperature case the energy equation is only dependent on pressure and volume without temperature dependence

\[
\frac{3}{2} \frac{1}{V^{2/3}} \frac{d}{dt} \left( PV^{5/3} \right) \approx P(t)^2 \int \Pi(T) dV \\
\Pi(T) \approx \frac{\varepsilon_\alpha}{16} S_f
\]

\[
\frac{1}{V^{2/3}} \frac{d}{dt} \left( PV^{5/3} \right) \approx \frac{\varepsilon_\alpha}{24} S_f P(t)^2 V(t) \\
\text{In 1D} \rightarrow V = \frac{4}{3} \pi R^3
\]

• In 1D, couple energy equation to Newton’s law for thin shell (shell momentum equation)

\[
M_{sh} \frac{d^2 R}{dt^2} = 4\pi R^2 P
\]
• Simplest DYNAMIC model of a thin shell compressing the hot spot in the high temperature limit where alpha heating exceeds rad losses and $\langle \sigma v \rangle \sim T^2$

• Two equations for the two unknowns $R(t)$ and $P(t)$

$$M_{sh} \frac{d^2 R}{dt^2} = 4\pi R^2 P$$

$$\frac{d}{dt}(PR^5) \approx \frac{\varepsilon_\alpha}{24} S_f P^2 R^5$$

• Initial conditions at the time $t=0$ of peak implosion velocity ($V_i$). Starting from that time the shell stops being accelerated and starts coasting and decelerating. The radius at the time of peak velocity is denoted as $R_*$

$$R(0) = R_*$$

$$\frac{dR}{dt}(0) = -V_i$$