The R-B instability develops when a fluid in a gravitational field has a temperature gradient in the direction of the gravitational acceleration.

It is well known that the warmer fluid in the bottom is lighter than the cold fluid at the top. Thus, the buoyancy force pushes the warm fluid up and the cold fluid down. As the warm fluid reaches the top, it cools down and becomes the cold fluid which in turns moves back down. Such process is called "natural convection".

---

Diagram:

- Hot fluid rising
- Cold fluid descending
- Arrows indicating direction of movement due to natural convection

---
Equation of motion for a liquid with a temperature gradient.

Mass: \[ \sum F + \nabla \cdot (\rho \nabla \mathbf{v}) = 0 \] ①

Momentum: \[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \rho \nabla \Phi + \nabla \cdot \mathbf{\tau} \] ②

Temp.: \[ \rho c_v \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \kappa \nabla^2 T - \rho \mathbf{v} \cdot \nabla \mathbf{v} + \Phi \] ③

where \( \Phi \) is the viscous heating and \( \mathbf{\tau} \) is the viscous stress tensor.

\[ \nabla \cdot \mathbf{\tau} = \frac{2}{\partial \mathbf{\tau}_{ij}} \] ④

The "equation of state" relating density and temperature is

\[ \rho = \rho_0 \left[ 1 + \alpha (T_0 - T) \right] . \] ④
where $p_0$ and $T_0$ are pressure and temperature at the lower boundary.

Equation (4) can be derived as a Taylor expansion of a more general equation of state

$$f = f(P, T)$$

The Taylor expansion about a given state $P_0, T_0, f_0$ yields

$$f = f_0 + \left( \frac{\partial f}{\partial P} \right)_0 (P - P_0) + \left( \frac{\partial f}{\partial T} \right)_0 (T - T_0)$$

Because liquids are almost incompressible then $\frac{\partial f}{\partial P} \approx 0$ and

$$f \approx f_0 + \left( \frac{\partial f}{\partial T} \right)_0 (T - T_0) = f_0 (1 + \alpha (T_0 - T))$$

where $\alpha = -\frac{1}{f_0} \left( \frac{\partial f}{\partial T} \right)_0$ is the expansion coefficient of the liquid.
Typically $\alpha$ is a very small parameter as liquid densities change little with temperature ($\alpha \approx 10^{-3} - 10^{-4} / \text{c}^2$ so the liquid density changes by $\approx 1\%$ for a $10^{\circ}$ variation of temperature).

Let's inspect the equation of motion:

1. \[ \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\nabla p \]

Since \[ p = p_0 (1 + \alpha (T_0 - T)) \], the left hand side of (1) is proportional to $\alpha$ while the RHS isn't.

Since $\alpha$ is small we can neglect the LHS and conclude that $\nabla \cdot \mathbf{v} = 0$
Therefore, since $\nabla \cdot \mathbf{v} \approx 0$, the Eq. (9) becomes:

$$p_0 (1 + \alpha \Delta T) \left( \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + p_0 (1 + \alpha \Delta T) \ddot{\mathbf{g}} +$$

$$+ \frac{1}{\sigma_{ij}} \left( \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} \mu \nabla \cdot \mathbf{v} \delta_{ij} \right)$$

This term can be rewritten as $\mu \nabla^2 \mathbf{v}$ (see page 13 of lecture #2).

Simplified momentum equation:

$$p_0 \left( \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + p_0 (1 + \alpha (T - T_0)) \ddot{\mathbf{g}} + \mu \nabla^2 \mathbf{v}$$

Let's consider the temperature equation:

$$p_0 (1 + \alpha \Delta T) c_v \left( \partial_t T + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T -$$

small because $\alpha$ is small and $\Delta T$ is small

$$-p \nabla \cdot \mathbf{v} + \dot{\phi}$$ small because $\phi \approx v^2$.
Final set (Boussinesq approximation)

\[ \nabla \cdot \mathbf{v} = 0 \quad (1) \]

\[ p_0 (\rho e \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + p_0 (1 + \alpha (T_0 - T)) \mathbf{g} + \mu \nabla^2 \mathbf{v} \quad (2) \]

\[ p_0 \rho e \nabla (\rho e T + \mathbf{v} \cdot \nabla T) = K \nabla^2 T \quad (3) \]

\[ \frac{dp}{dz} = -p_0 (1 + \alpha (T_0 - T)) g \]

\[ \frac{d^2 T}{dz^2} = 0 \]

Consider a fluid heated from below and at rest.
\[ T = T_0 - z \beta \quad \text{\( \beta = -\frac{dT_0}{dz} \) is the temp. gradient.} \]

\[ \frac{dp}{dz} = -p_0 (1 + \alpha \beta z) g \]

\[ p = p_0 - g p_0 \left( z + \alpha \beta z^2 \right) \]
STABILITY

Linearization:

\( \mathbf{V} \cdot \mathbf{\tilde{V}} = 0 \)  \( \text{4} \)

\( \rho_x \mathbf{\tilde{V}} = - \nabla p + \rho_0 x T g \mathbf{\hat{e}_z} + \mu \nabla^2 \mathbf{\tilde{V}} \)  \( \text{5} \)

\( \rho_0 c_v \left( \mathbf{\tilde{T}} + \mathbf{\tilde{e}_z} \mathbf{\tilde{Q}_z T} \right) = K \nabla^2 \mathbf{\tilde{T}} \)  \( \text{6} \)

Use \( \mathbf{\tilde{Q}_z T} = -B \) and \( \mathbf{\tilde{D}_T} = \frac{K}{\rho_0 c_v} \). Rewrite \( \text{6} \) as

\( \mathbf{\tilde{T}} - \mathbf{\tilde{e}_z} B = \mathbf{\tilde{D}_T} \nabla^2 \mathbf{\tilde{T}} \)  \( \text{6'} \)

Take \( \nabla \times (5) \)

\( \rho_0 x \mathbf{\nabla} \times \mathbf{\tilde{V}} = \rho_0 x g \mathbf{\nabla} \times \mathbf{\hat{e}_z} + \mu \nabla^2 (\nabla \mathbf{\times \tilde{V}}) \)  \( \text{7} \)
Take $\nabla \times (\hat{r})$.

\[
\rho_0 \times \nabla \times (\nabla \times \hat{r}) = \rho_0 \alpha g \nabla \times (\nabla \times \hat{r}) + \mu \nabla^2 \nabla \times (\nabla \times \hat{r}) \tag{8}
\]

Use the identity

\[
\nabla \times \nabla \times \hat{r} = \nabla (\nabla \cdot \hat{r}) - \nabla^2 \hat{r} \approx -\nabla^2 \hat{r}.
\]

\[
\nabla \times (\nabla \times \hat{r}) = -\hat{e}_z \nabla^2 \hat{r} + (\hat{e}_z \cdot \nabla) \nabla \hat{r}.
\]

Rewrite (8) as

\[
-\rho_0 \times \nabla^2 \hat{r} = \rho_0 \alpha g \left[ \nabla \frac{\partial \hat{r}}{\partial z} - \hat{e}_z \nabla^2 \hat{r} \right] - \mu \nabla^2 (\nabla^2 \hat{r}) \tag{9}
\]

Take the $z$ component of (9)

\[
-\rho_0 \times \nabla^2 \hat{r}_z = -\rho_0 \alpha g \left( \frac{\partial \hat{r}_z}{\partial x^2} + \frac{\partial \hat{r}_z}{\partial y^2} \right) - \left( \frac{\mu}{\rho_0} \right) \nabla^4 \hat{r}_z \tag{10}
\]

$\hat{r} = \text{Kinematic Viscosity}$
Equations 10 and 6 represent two eqs. with two unknowns.

\[(f - D_T \nabla^2) \tilde{T} - \beta \tilde{u}_z = 0.\]  

\[\nabla^2 (\gamma - \nu \nabla^2) \tilde{u}_z - \alpha g (\tilde{\omega}_{xx} + \tilde{\omega}_{yy}) \tilde{T} = 0.\]  

Use Fourier decomposition in \(x\) and \(y\).

\[\tilde{Q} = \tilde{Q}_e \exp(i k_x x + i k_y y).\]

Rewrite 11 and 12.

\[(f + D_T k^2 - D_T \tilde{\omega}_{zz}) \tilde{T} - \beta \tilde{u}_z = 0.\]

\[(-k^2 + \tilde{\omega}_{zz}) (\gamma + k^2 \nu - \nu \tilde{\omega}_{zz}) \tilde{u}_z + \alpha g k^2 \tilde{T} = 0.\]

where \(k^2 = k_x^2 + k_y^2\).
Use dimensionless variables:

\[ \tilde{y} = \frac{y}{d}, \quad \tilde{z} = \frac{z}{d}, \quad \tilde{\sigma} = \frac{\sigma d^2}{\nu}, \quad \tilde{P}_r = \frac{\nu}{d^3} \]

\[ \text{Rewrite (13) and (14).} \]

\[ \left( \frac{\tilde{\sigma} \tilde{v}}{d^2} + \frac{D_r}{\nu d^2} \frac{\tilde{d}^2}{d^2} - \frac{D_r}{\nu d^2} \tilde{\sigma}_{33} \right) \tilde{T} - \beta \tilde{V}_z = 0. \]  

\[ \left( \frac{-\tilde{a}^2 + \frac{1}{4} \tilde{\sigma}_{33}}{d^2} \right) \left( \frac{\tilde{\sigma} \tilde{v}}{d^2} + \frac{\tilde{v}^2}{d^2} - \frac{\nu}{d^2} \tilde{\sigma}_{33} \right) \tilde{V}_z + \frac{\alpha g d^2}{\nu} \tilde{T} = 0. \]

\[ \left( \frac{\tilde{a}^2 - \sigma}{D_r} \right) \left( \tilde{a}^2 - \sigma \tilde{P}_r \right) \tilde{T} = -\left( \frac{\beta}{D_r} \frac{d^2}{\nu} \right) \tilde{V}_z. \]

\[ \left( \tilde{a}^2 - \frac{\sigma}{\tilde{P}_r} \right) \left( \tilde{a}^2 - \frac{\sigma}{\tilde{P}_r} \right) \tilde{V}_z = \left( \frac{\alpha g d^2}{\nu} \right) \frac{d^2}{\nu} \tilde{T}. \]

The next step is to solve Eqs. (17) and (18) and apply the Boundary conditions.
Boundary Conditions

The simplest case (but also the least interesting from the experimental point of view) is the one where both the top and bottom surfaces are fixed but no friction is present. In addition, both surfaces are maintained at constant temperature. In this case, the boundary conditions require that no tangential stress are applied on the surfaces.

That is: $T_{xz} = 0$, $T_{yz} = 0$.

$$T_{xz} = \mu \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = 0, \quad T_{yz} = \mu \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) = 0$$

Furthermore, a fixed surface requires that $u_z = 0$ on the top and bottom.

It follows that $T_{xz} \mid_{0,d} = \mu \frac{\partial u_x}{\partial z} = 0$.

$$T_{yz} \mid_{0,d} = \mu \frac{\partial u_y}{\partial x} = 0.$$
Using the incompressibility condition, it follows that
\[ \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \]
and
\[ \frac{\partial}{\partial x} \left( \frac{\partial u_x}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u_y}{\partial z} \right) + \frac{\partial u_z}{\partial z^2} = 0. \]

It follows that \( \frac{\partial^2 u_z}{\partial z^2} = 0 \) at \( z = 0, d \).

Summary of B.C. for free surface

\[ \begin{cases} v_z = 0 \\ \frac{\partial^2 u_z}{\partial z^2} = 0 \end{cases} \text{ at } z = 0 \text{ and } z = d. \]
Back to eqs. (17) and (18).

Eq. (18) can be rewritten as

\[
0^4 \frac{\partial}{\partial z} \left( \frac{1}{2} \alpha^2 \sigma \right) 0_{33} \tilde{u}_z - a^2 0_{33} \tilde{u}_z + a^2 (a^2 + \sigma) \tilde{u}_z = \left( \frac{\alpha \beta d}{\nu} \right) a^2 \tilde{u}_z
\]

On the boundaries \( 0_{33} \tilde{u}_z = 0, \tilde{u}_z = 0, \tilde{T} = 0, \)

therefore \( 0^4 \frac{\partial}{\partial z} \tilde{u}_z = 0 \) at \( z = 0, d \) (or \( z = 0, 1 \))

Taking the \( 0_{33} \) of (19) it is easy to show that \( 0^6 \frac{\partial}{\partial z} \tilde{u}_z = 0 \) at \( z = 0, d \) and therefore

\[
0^m \frac{\partial}{\partial z} \tilde{u}_z = 0 \text{ at } z = 0, 1 \text{ for } m = 0, 1, 2, \ldots
\]

Eliminating \( \tilde{T} \) between (17) and (18) leads to

\[
\left( 0_{33} - a^2 \right) \left( 0_{33} - a^2 - \sigma \right) \left( 0_{33} - a^2 - \sigma Pr \right) \tilde{u}_z = -\frac{\alpha \beta d}{\nu} a^2 \tilde{u}_z
\]

with B.C. \( 0^m \frac{\partial}{\partial z} \tilde{u}_z = 0 \) at \( z = 0, 1 \)
Eq. (20) is a 6th-order ODE with constant coefficients. Therefore, its solutions are in the form of exponentials $e^{\alpha z}$.

It is clear that the only solutions made up of exponentials which satisfy the boundary condition $\tilde{\phi}_z(0) = 0$ at $z = 0, 1$ is

$$\tilde{\phi}_z = \sin(m\pi z)$$

where $m$ is an integer.

Substituting into (20) yields

$$\left(m^2 \pi^2 + \alpha^2\right) \left(m^2 \pi^2 + \alpha^2 + \sigma^2\right) \left(m^2 \pi^2 + \alpha^2 + 8 \Pr \right) = \frac{\alpha \sigma}{\nu D} \beta^4 \sigma^2 \frac{d^4}{\nu D \tau}$$

The quantity $\frac{\alpha \sigma}{\nu D \tau}$ is dimensionless and referred to as the Rayleigh number

$$R = \frac{\alpha \sigma}{\nu D \tau}$$
Eq. (2) is rewritten as

\[ R = \frac{\left( m^2\pi^2 + a^2 \right) \left( m^2\pi^2 + a^2 + \sigma^2 \right) \left( m^2\pi^2 + a^2 + \sigma^2 + \delta^2 \right)}{a^2} \]

The marginal stability \((\lambda = 0 = \sigma = 0 \text{ state})\) occurs for Rayleigh numbers

\[ R = \frac{\left( m^2\pi^2 + a^2 \right) \left( m^2\pi^2 + a^2 \right)}{a^2} = \left( m^2\pi^2 + a^2 \right)^2 \]

The lowest \(R\) occurs for \(m = 1\).

\[ R = \frac{(\pi^2 + a^2)^3}{a^2} \]

And the minimum of the lowest \(R\) occurs for

\[ \frac{dR}{da^2} = 0 = 3 \frac{(\pi^2 + a^2)^2 \cdot 2a^2 - (\pi^2 + a^2)^3}{a^2} = 0 \]

\[ 2a^2 = \pi^2 \quad a^2 = \frac{\pi^2}{2} \]
\[ R_{\text{critical}} = R \left( \frac{m=1}{a^2 = \pi^2} \right) = \pi^4 \left( \frac{1 + \frac{1}{2}}{\frac{1}{2}} \right)^{\frac{3}{2}} = 3 \pi^4 \times \frac{4}{4} = 657.5. \]

This is the critical Rayleigh number. For \( R > R_{\text{critical}} = 657.5 \), the equilibrium is unstable and a convective motion is set up. The critical wavenumber destabilizes when \( R = R_{\text{critical}} \) is \( \omega^2 = K_u d^2 = \frac{\pi^2}{2} \).