The effect of a magnetic field on hydrodynamic instabilities.

In the sun, the magnetosphere or in laboratory plasmas, magnetic fields are often present. The effect of magnetic fields on the evolutions of hydro-instabilities can be quite dramatic.

First let's consider the problem of how a magnetic field affects the wave propagation in a uniform medium. This is useful as it helps to understand the physics of the interaction between electrically conducting fluids and magnetic fields.

Consider a uniform fluid with constant electrical resistivity $\eta$.

Ohm's law relates the electric field $\vec{E}$ and the current density $\vec{J}$:

$$\vec{E} = \eta \vec{J}$$

Ohm's law in the fluid frame of reference.

The fluid velocity $\vec{v} = \vec{E} + \vec{J} \times \vec{B}$ in the Lab frame of reference. (See also Appendix A).
The electric and magnetic fields are related through Faraday’s law:
\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad \text{Faraday’s law}
\]

Faraday’s law indicates that a time-varying magnetic field generates an electric field. Magnetic fields are generated by currents and time-varying electric fields as indicated by Ampere’s law:
\[
\mu_0 \mathbf{J} = \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{Ampere’s law}
\]

Typically, the second term on the RHS is small unless the electric field changes in time on a time scale of the light propagation through the system.

Thus, the so-called “low-frequency” Ampere’s law is frequently used,
\[
\mu_0 \mathbf{J} = \nabla \times \mathbf{B}
\]
The magnetic field acts on the conducting fluid through the Lorentz force.

A charged particle (with charge $q$) travelling with velocity $\vec{v}$ through a magnetic field $\vec{B}$ is subject to a force

$$\vec{F} = q\vec{v} \times \vec{B}$$

Since an electric current density is made up of the electrons crossing the unit surface per unit time,

$$\vec{J} = -ne\vec{E}$$

than the force applied to the unit volume is simply

$$-me\vec{v} \times \vec{B} = \vec{J} \times \vec{B} \quad \text{Lorentz force}$$

force applied to a single electron.
It follows that the momentum equation for a conducting fluid must be modified using the Lorentz force.

\[
\mathbf{p} \frac{d\mathbf{v}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} + \nabla \cdot \mathbf{\tau} + \mathbf{p} \cdot \hat{\mathbf{g}}
\]

Viscous tensor = \((\mu \nabla^2)\) for incompressible fluids.

In summary, the "magneto-hydrodynamic" equations describing a conducting fluid are:

1. \( \nabla \cdot \mathbf{p} + \nabla \cdot (\mathbf{p} \mathbf{v}) = 0 \)

2. \( \mathbf{J} \cdot (\nabla \cdot \mathbf{v}) = -\nabla p + \mathbf{j} \times \mathbf{B} + \mathbf{p} \cdot \hat{\mathbf{g}} + \nabla \cdot \mathbf{\tau} \)

3. \( \rho c_v (\nabla \cdot \mathbf{v}) + \rho c_v (\nabla \cdot \mathbf{v}) = -p (\nabla \cdot \mathbf{v}) + 2 \mathbf{j}^2 + \Phi + \nabla \cdot \mathbf{\tau} \)

4. \( \rho = \rho(p, \mathbf{v}, T) \) - equation of state.

5. \( \nabla \cdot \mathbf{B} = 0 \) - solenoidal field.

6. \( \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \)

7. \( \mathbf{E} + \mathbf{j} \times \mathbf{B} = \nabla \phi \)

8. \( \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \)

Low frequency

Maxwell's equations.
where (1) is the mass conservation,
(2) is the momentum conservation,
(3) is the temperature (or thermal energy equation) including the Joule heating $2J^2$ where $J$ is the fluid resistivity, $\Phi$ is the viscous heating and $K$ is the thermal conductivity.

Eq. (4) is the equation of state.
Eq. (5) is the low-frequency Amperere's law.
Eq. (6) is the Ohm's law.
Eq. (7) is the Faraday's law.
Waves in a conducting fluid.

Considers an homogeneous, infinite conducting fluid in a straight uniform, magnetic field.

No gravity, no viscosity, no resistivity - an ideal superconducting fluid.

For simplicity, take the fluid to be incompressible.

**IDEAL INCOMPRESSIBLE MHD (Magnetohydrodynamics)**

\[ \rho = \text{const.} \quad \nabla \cdot \vec{v} = 0 \]

\[ \rho (\vec{c} \cdot \nabla \vec{v} + \vec{v} \cdot \nabla \vec{v}) = -\nabla p + \dot{\vec{J}} \times \vec{B} \]

\[ \vec{E} + \vec{c} \times \vec{B} = 0 \]

\[ \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} \]

\[ \dot{\vec{J}} = \nabla \times \vec{B} \]

\[ \nabla \cdot \vec{B} = 0 \]
Equilibrium:
\[ p = \text{const} \]
\[ \nabla \times B = 0 \]
\[ \delta^2 = 0 \]
\[ E^2 = 0 \]
\[ J^2 = 0 \]
\[ \nabla p = 0 \]

Perturbation analysis:
\[ \rho \tilde{V} = -\nabla \tilde{p} + \tilde{\omega} \times B + \underbrace{\tilde{\omega} \times \tilde{\omega}}_{= 0} \]
\[ \tilde{E} = -\tilde{\omega} \times B \]
\[ \tilde{\omega} \times \tilde{E} = -\nabla \times \tilde{E} \]
\[ \mu_0 \tilde{J} = \nabla \times \tilde{B} \]
\[ \nabla \cdot \tilde{V} = 0, \quad \nabla \cdot \tilde{B} = 0. \]

Since the equilibrium is uniform, we can

Fourier analyze in all 3 directions:
\[ \tilde{Q} = \tilde{Q} e^{st + i K \cdot x} \]
\[ x = (x \hat{e}_x + y \hat{e}_y + z \hat{e}_z) \]
\[ K = (K_x \hat{e}_x + K_y \hat{e}_y + K_z \hat{e}_z) \]
Linearized equations:

\( \vec{\nabla} \vec{V} = -i \vec{k} \vec{\varphi} + \vec{\nabla} \times \vec{B} \)

\( \vec{E} = \vec{\nabla} \times \vec{B} \)

\( \vec{B} = -i \vec{k} \times \vec{E} \)

\( \mu_0 \vec{J} = i \vec{k} \times \vec{B} \)

\( i \vec{k} \cdot \vec{V} = 0 \)

Take \( \vec{k} \times (8) \rightarrow \vec{\nabla} \vec{k} \times \vec{V} = + \vec{k} \times (\vec{J} \times \vec{B}) \)

Use (11) into (13) \( \rightarrow \vec{\nabla} \vec{k} \times \vec{V} = \vec{k} \times \frac{[\vec{k} \times \vec{B} \times \vec{B}]}{\mu_0} \)

Use (10) into (14)

\( \vec{\nabla} \vec{k} \times \vec{V} = \vec{k} \times \left[ \frac{(\vec{k} \times \vec{B} \times \vec{B})}{\mu_0} \right] \)

Use (9) into (15)

\( \vec{\nabla} \vec{k} \times \vec{V} = -\vec{k} \times \left[ \left( \vec{k} \times \left( \vec{k} \times (\vec{B} \times \vec{B}) \right) \right) \times \vec{B} \right] \)
Carry out the cross product using the identity

\[ \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \mathbf{C} \]

\[ \mathbf{K} \times (\mathbf{\hat{a}} \times \mathbf{B}) = (\mathbf{K} \cdot \mathbf{\hat{a}}) \mathbf{B} - (\mathbf{K} \cdot \mathbf{B}) \mathbf{\hat{a}} \]

\[ \mathbf{K} \times (\mathbf{\hat{a}} \times \mathbf{B}) = 0 \] because of (12).

Eq. (16) becomes:

\[ \rho \dot{\mathbf{k}} \times \mathbf{\hat{a}} = - \mathbf{k} \times [\mathbf{k} \times \mathbf{\hat{a}}] \mathbf{B} \left( \mathbf{K} \cdot \mathbf{B} \right) / \mu_0 \]

\[ \rho \dot{\mathbf{k}} \times \mathbf{\hat{a}} = - \mathbf{k} \times \mathbf{\hat{a}} \cdot (\mathbf{k} \cdot \mathbf{B})^2 / \mu_0 \] (17).

Eq. (17) yields

\[ \begin{cases} \rho \mathbf{k} \times \mathbf{\hat{a}} = 0 & \text{or} \\ \rho \dot{\mathbf{k}} \times \mathbf{\hat{a}} = - (\mathbf{k} \cdot \mathbf{B})^2 / \mu_0 \end{cases} \] (18).

If (18) is true then combining (18) and (12)
yields \( \mathbf{\hat{a}} = 0 \) which is the trivial solution of zero perturbation.

The non-trivial solution requires

\[ \rho^2 = - \left( \frac{(\mathbf{k} \cdot \mathbf{B})^2}{\mu_0} \right) \Rightarrow \rho = \pm i \omega \]

\[ \omega = \left( \frac{(\mathbf{k} \cdot \mathbf{B})^2}{\mu_0} \right) / \rho \]

Oscillatory mode (wave)
The quantity $\sqrt{\frac{B^2}{\mu_0 \rho}}$ has the dimension of the velocity and is referred to as Alfven velocity, $V_A$.

The quantity $\sqrt{\frac{(\mathbf{k} \cdot \mathbf{B})^2}{\mu_0 \rho \mu}} = \frac{k_|| B}{\sqrt{\mu_0 \rho}}$ is the Alfven frequency.

where $k_||$ is the component of the wavenumber in the direction of the magnetic field.

The oscillatory modes with frequency

$\omega = \frac{k_|| B}{\sqrt{\mu_0 \rho}}$

are the Alfven waves. These waves are caused by the fact that magnetic field lines behave like rubber bands in superconducting fermals ($\eta = 0$).

Consider a perturbation with $\mathbf{K} = k_|| \mathbf{b}$

($\mathbf{b}$ is the unit vector in the $\mathbf{B}$ direction).
Since \( \mathbf{K} \cdot \mathbf{V} = 0 \) then \( V_{\parallel} = 0 \) and \( \mathbf{V} \) is perpendicular to \( \mathbf{B} \).

It follows that such waves distort the magnetic field lines.

\[ \text{wavelength } \lambda = \frac{2\pi}{k_{\parallel}} \]

perturbation behaves as \( e^{-i\omega t} \)

The field line oscillates like a rubber band.

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Rayleigh–Taylor instability in the presence of a magnetic field

Simple problem
Use the momentum equation in the form

\[ F(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \mathbf{j} \times \mathbf{B} \]

Replace \( \mu_0 \mathbf{j} = \nabla \times \mathbf{B} \)

Thus

\[ F(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla (p + \frac{B^2}{2\mu_0}) + \left(\frac{\mathbf{B} \cdot \mathbf{v}}{\mu_0}\right) \mathbf{B} \]

**Equilibrium**

\[ p = \text{const.} \]

\[ J = 0 \text{ at } z > 0 \text{ and } z < 0 \quad (\text{but not at } z = 0) \]

\[ \frac{d}{dz} \left( p + \frac{B^2}{2\mu_0} \right) = -pg \]

\[ p + \frac{B^2}{2\mu_0} = -pgz + p_0 \]

For \( z > 0 \), \( B = 0 \) and \( p = p_0 - pgz \).

For \( z < 0 \), \( \nabla p = 0 \) and \( \frac{B^2}{2\mu_0} = p_0 - pgz \).

At \( z = 0 \), \( p_0 = \frac{B^2}{2\mu_0} \).
STABILITY

At the fluid-vacuum interface \( z = \mathcal{S}(x, t) \),

\[
\mathbf{B} \cdot \mathbf{m} = 0 \quad \text{where} \quad \mathbf{m} = \frac{(\mathbf{Q} \times \mathbf{s}) \hat{\mathbf{x}} - \hat{\mathbf{z}}}{\sqrt{1 + (\mathbf{Q} \cdot \mathbf{s})^2}}
\]

is the unit vector perpendicular to the interface.

In the vacuum \( \nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \nabla^2 \mathcal{S} = 0 \).

\( \nabla \times \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \mathcal{S} \)

then at the fluid-vacuum interface, the \( \mathbf{B} \) field satisfies,

\[
\mathbf{m} \cdot \nabla \mathcal{S} \propto (\mathbf{Q} \times \mathbf{s}) (\mathbf{Q} \times \mathbf{s}) - \mathbf{Q}_z \mathcal{S} = 0.
\]

Set of equations:

1) \( \nabla \cdot \mathbf{v} = 0 \)
2) \( f (\mathbf{Q}_x \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \rho \mathbf{g} \) in fluid where \( \mathbf{B} = 0 \)
3) \( \nabla^2 \mathcal{S} = 0 \) in the vacuum.
4) \( \mathbf{m} \cdot \nabla \mathcal{S} = 0 \Rightarrow (\mathbf{Q} \times \mathbf{s}) (\mathbf{Q} \times \mathbf{s}) - \mathbf{Q}_z \mathcal{S} = 0 \) at \( z = \mathcal{S}(x, t) \).
5) \( \mathbf{Q}_x \mathbf{v} + \nabla \times \mathbf{Q} \mathbf{s} = \mathbf{v}_z \) at \( z = \mathcal{S}(x, t) \) — interface equation.
Linearize the equations:

\( \nabla \cdot \mathbf{v} = 0 \quad \text{constant} \Rightarrow \mathbf{v} = 0 \)

\( \rho \nabla \cdot \mathbf{v} = -\nabla \mathbf{p} + \mathbf{R} = 0 \)

\( \nabla^2 \mathbf{v} = 0 \)

Equilibrium B.

\( \partial_z \mathbf{v} = 0 \quad \text{at} \quad z = 0 \quad \Rightarrow \quad \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{at interface} \)

\( \nabla^2 \mathbf{v} = \mathbf{v} \quad \text{at} \quad z = 0 \quad \Rightarrow \quad \text{interface condition} \)

Jump conditions = force balance across interface

\[ \left\| \mathbf{p} + \frac{\mathbf{B}^2}{2 \mu_0} \right\| = 0 \]

[One can get this also by integrating the moment eq. across interface]

\[ \left\| \left( \frac{\mathbf{p} + \frac{\mathbf{B} \cdot \mathbf{B}}{\mu_0}}{2} \right) + \frac{d}{dz} \left[ \frac{\mathbf{p} + \frac{\mathbf{B}^2}{2 \mu_0}}{2} \right] \right\|_{z=0}^{z=0} = 0 \]

\( -pg \) (from equilibrium relations).
Solve in the fluid \((z > 0)\) for rigid well at \(d\):

\[
\begin{align*}
\mathbf{v}_x &= -iKp/\rho, & \mathbf{v}_z &= i\frac{\partial \mathbf{v}_x}{\partial z}, & \mathbf{v}_z &= \hat{u} \sinh(K(z-d)) \\
\mathbf{p} &= \frac{i}{K} \rho \mathbf{v}_x
\end{align*}
\]

Solve in the vacuum:

\(\nabla^2 \mathbf{U} = 0 \Rightarrow \nabla^2 \mathbf{U} - K^2 \mathbf{U} = 0\) – Schwarz

Boundary conditions at bottom well \(z = -d\):

Assume that bottom well is superconducting so that \(\mathbf{B} \cdot \mathbf{n}_W = 0\) where \(\mathbf{n}_W = \hat{e}_z\) is the normal unit vector to the well.

Thus, \(\mathbf{v}_z \big|_{z = -d} = 0 \Rightarrow \mathbf{v}_z = \hat{z} \cosh[K(z+d)]\).

Apply B.C. at interface:

\(\mathbf{v}_z = \hat{u} \sinh(Kd)\) – interface eq.

\(\mathbf{B} = K \hat{e}_z \sinh(Kd) = 0\) – \(\mathbf{B} \cdot \mathbf{n} = 0\).

\[-\frac{i}{K^2} \rho \mathbf{g} = \frac{K \hat{u} \cosh(Kd)}{\mu_0}\] – force balance.
Rearrange force balance using the other two equations:

$$\frac{1}{k^2} \gamma \times \frac{\gamma}{\frac{\gamma}{\gamma} \cosh(kd)} - \frac{pg}{\gamma} = iB \cosh(kd) \frac{iB^2}{\mu_0} \frac{\gamma}{\gamma} \cosh(kd)$$

$$\gamma^2 = Kg \tanh(kd) - \frac{k^2 B^2}{\mu_0 \rho}$$

Remember that $\sqrt{B^2/\mu_0}$ is the Alfvén velocity $V_A$ and that $KV_A = \omega_A$ is the Alfvén frequency.

$$\gamma = \sqrt{Kg \tanh(kd) - k^2 V_A^2}$$

Observe that the magnetic field is stabilising.

Its contribution is proportional to $k^2$ and gets larger at short wavelengths.
This effect can be easily explained by the fact that magnetic field lines behave as rubber bands.

\[ \text{Fluid} \]
\[ \text{Vacuum} \]
\[ \text{B field} \]

Since the fluid is superconducting, the magnetic field can't penetrate the fluid, and the growing distortion of the interface is forced to bend the magnetic field lines. The shorter the wavelength, the larger the bending, and the greater is the restoring force of the B-field lines.

Also observe from the dispersion relation that a cut-off wavenumber exist

\[ k_c \approx \frac{1}{\pi} \left( \frac{V_o}{d} \right) \]

For \( k \geq k_c \) the instability is suppressed.
So far, we have considered perturbations behaving as \( e^{ikx} \), where \( k \) is the direction of the field. Let's now consider the general case with
\[
\hat{A} = \hat{A} \cdot e^{ikx + iky}.
\]

**Linearized Equations.**

1) \( \nabla \cdot \bar{\psi} = 0 \Rightarrow \bar{\psi} = -D_e \bar{\psi}_2 \quad k = k_x \hat{e}_x + k_y \hat{e}_y \)

2) momentum \( \gamma D_\xi \bar{\psi}_1 = -i \bar{\psi} \quad \gamma D_\xi \bar{\psi}_2 = -D_e \bar{\psi} \quad \text{massless} \)

3) vacuum \( \nabla^2 \bar{\psi} = 0 \Rightarrow D_{zz} \bar{\psi} - k^2 \bar{\psi} = 0 \quad k^2 = k_x^2 + k_y^2 \)

Combine 1) and 2):

\[
-\gamma D_\xi \bar{\psi}_2 = k^2 \bar{\psi}_2 \Rightarrow \bar{\psi} = -\frac{\gamma D_\xi}{k^2} D_{zz} \bar{\psi}_2.
\]

\[
\rho \gamma \bar{\psi}_2 = -\bar{\psi} \bar{\psi} = \frac{\rho \gamma}{k^2} D_{zz} \bar{\psi}_2 \Rightarrow D_{zz} \bar{\psi}_2 - k^2 \bar{\psi}_2 = 0.
\]

\[\bar{\psi}_2 = \hat{\psi} \sinh[k(x + y)] \quad \text{as before}.
\]
Rewrite jump conditions and B.C. at interface.

4) \( \frac{\hat{V}_z}{\hat{V}_0} = -\hat{U} \sinh(Kd) \) — interface equation.

5) \( \sum B \hat{E}_z - \sum E_z = iK_x B \hat{S}_z - \hat{S}_x \hat{K} \sinh(Kd) = 0 \) — \( \sum B \cdot \hat{M} = 0 \) at interface.

6) \( -\frac{\hat{\gamma}}{K} \hat{U} \cosh(Kd) - pg \hat{S}_z = \frac{iK_x B \hat{S}_z}{\mu_0} \cosh(Kd) \) — force balance.

Combining 4), 5), and 6) yields:

\[
\frac{\hat{\gamma}}{K} \hat{U} \frac{1}{\sinh(Kd)} \cosh(Kd) - pg \frac{1}{\hat{S}_x} = \frac{iK_x B \cosh(Kd)}{\mu_0} \frac{1}{K \sinh(Kd)} \hat{S}_z
\]

\[
\text{DISPERSION RELATION}
\]

\[
\hat{\gamma}^2 = Kg \tanh(Kd) - \frac{K_x B^2}{\mu_0 \hat{p}}
\]

\[
\gamma^2 = Kg \tanh(Kd) - K_x \hat{v}_0^2
\]

where \( K = \sqrt{K_x^2 + K_y^2} \).
This is a very important result because it shows that any perturbation with $k_x = 0$ and $k_y \neq 0$ (i.e. $\mathbf{q} = \hat{z} e^{i k_y y}$) would not be affected by the magnetic field as

$$\mathbf{\gamma}(k_x = 0) = k_y g \tanh(k_y d)$$

Observe that $k_x = 0$ means that the perturbation is constant along the magnetic field lines, and therefore it does NOT bend the magnetic field. Instead, the variation in $y$-only shows that the perturbation displaces the field lines in the $y$-direction without bending them.

[Diagram of magnetic field lines with perturbation shown]
The perturbation has $k_x = 0$, $k_y \neq 0$. No bending occurs.

View of R-T instability for $k_x = 0$, $k_y \neq 0$.

Since, no bending occurs when $k_x = 0$, the instability is not stabilized by the magnetic field and grows as in the unmagnetized medium:

$$\sigma = \sqrt{k_y g \tanh(k_y d)}.$$
If the fluid is superconducting, then

\[ \phi = \text{constant over a surface moving with the fluid}, \]

where \( \phi = \int_{S} \mathbf{\hat{B}} \cdot \mathbf{\hat{m}} \, ds \) and \( S \) is a surface moving with the fluid.

Consider the surface \( S \) that at time \( t \)
we call \( \Sigma \), and at time \( t + \Delta t \) is moved into the surface \( \Sigma_1 \).

\[ \Delta \phi = \int_{\Sigma_1} \mathbf{\hat{B}}(t + \Delta t) \cdot \mathbf{\hat{m}} \, ds - \int_{\Sigma} \mathbf{\hat{B}}(t) \cdot \mathbf{\hat{m}} \, ds. \]

Rewrite \( \Delta \phi \):

\[ \Delta \phi = \int_{\Sigma_1} \mathbf{\hat{B}}(t + \Delta t) \cdot \mathbf{\hat{m}} \, ds - \int_{\Sigma} \mathbf{\hat{B}}(t) \cdot \mathbf{\hat{m}} \, ds \]

\[ + \int_{\Sigma} \left( \mathbf{\hat{B}}(t + \Delta t) - \mathbf{\hat{B}}(t) \right) \cdot \mathbf{\hat{m}} \, ds. \]
Define with $\Sigma$ the surface described by the contour of $\Sigma^*$.

Rewrite:

$$\Delta \phi = \oint\limits_{\Sigma_1} \hat{\mathbf{B}}'(t+\Delta t) \cdot \hat{\mathbf{n}} \, d\Sigma - \int\limits_{\Sigma} \hat{\mathbf{B}}'(t+\Delta t) \cdot \hat{\mathbf{m}} \, d\Sigma$$

$$+ \int\limits_{\omega} \hat{\mathbf{B}}(t+\Delta t) \cdot \hat{\mathbf{m}} \, d\omega + \int\limits_{\Sigma} \left( \hat{\mathbf{B}}'(t+\Delta t) - \hat{\mathbf{B}}'(t) \right) \cdot \hat{\mathbf{m}} \, d\Sigma$$

$$- \int\limits_{\omega} \hat{\mathbf{B}}(t+\Delta t) \cdot \hat{\mathbf{m}} \, d\omega.$$

The first three terms represent the integral over the volume enclosed by $\Sigma, \Sigma_1, \omega$.

Using Gauss' theorem:

$$\Delta \phi = \int\limits_{V(t+\Delta t)} \left( \nabla \times \hat{\mathbf{B}} \right) \, dV + \int\limits_{\Sigma} \left( \hat{\mathbf{B}}'(t+\Delta t) - \hat{\mathbf{B}}'(t) \right) \cdot \hat{\mathbf{m}} \, d\Sigma$$

$$- \int\limits_{\omega} \hat{\mathbf{B}}'(t+\Delta t) \cdot \hat{\mathbf{m}} \, d\omega.$$
Since the surface $S$ moves with the fluid from $\Sigma$ to $\Sigma_1$, then the surface $\mathcal{W}$ is described by the fluid motion:

\[ \Delta \phi = \int_\Sigma \left( \hat{\mathbf{B}}(t+\Delta t) - \hat{\mathbf{B}}(t) \right) \cdot \hat{\mathbf{n}} \, dS - \int_\Sigma \hat{\mathbf{B}}(t+\Delta t) \cdot (\hat{\mathbf{e}} \times \hat{\mathbf{v}}) \, dS \Delta t \]

and

\[ \hat{\mathbf{B}} \cdot \hat{\mathbf{e}} \times \hat{\mathbf{v}} = \hat{\mathbf{\nabla}} \times \hat{\mathbf{B}} \cdot \hat{\mathbf{e}} \Delta t . \]

\[
\left[ \frac{\Delta \phi}{\Delta t} \right]_{\Delta t \to 0} = \frac{d\phi}{dt} = \int_\Sigma \hat{\mathbf{\nabla}} \cdot \hat{\mathbf{m}} \, dS - \int_\Sigma (\hat{\mathbf{\nabla}} \times \hat{\mathbf{B}}) \cdot \hat{\mathbf{e}} \, dS
\]

Use Stokes' theorem:

\[ \int_\Sigma \hat{\mathbf{A}} \cdot d\mathbf{e} = \int_\Sigma (\hat{\mathbf{\nabla}} \times \hat{\mathbf{A}}) \cdot \hat{\mathbf{m}} \, dS \]

Then:

\[ \frac{d\phi}{dt} = \int_\Sigma \left( \frac{\partial \hat{\mathbf{B}}}{\partial t} - \hat{\mathbf{\nabla}} \times (\hat{\mathbf{\nabla}} \times \hat{\mathbf{B}}) \right) \cdot \hat{\mathbf{m}} \, dS . \]
This should apply for arbitrary $\mathbf{E}$ so that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{v}' \times \mathbf{B} \right)$$

Since $\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times (\mathbf{E}')$ then

$$\mathbf{E}' + \mathbf{v}' \times \mathbf{B} = 0$$

← Ohm's law for superconducting fluids.

For a fluid with finite resistivity

$$\mathbf{E}' + \mathbf{v}' \times \mathbf{B} = \mathbf{\sigma} \mathbf{J}$$

← Ohm's law

resistivity