Compressible R-T instability
of two superimposed inviscid fluids

If the fluid density is not constant (such as in liquids), the assumption \( \nabla \cdot \mathbf{v} = 0 \) may not be right. We then replace the incompressibility condition (\( \nabla \cdot \mathbf{v} = 0 \)) with the full energy equation. We consider an ideal gas (\( p = \rho R T \)) and rewrite the internal energy in terms of its temperature.

\[
dU = C_v \, dT
\]

\( C_v = \frac{R}{\Gamma - 1} \quad \rho = \frac{C_p}{C_v} \)

You have shown in hw #1 that (neglect external sources)

\[
C_v \rho \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = - \rho \nabla \cdot \mathbf{v}^2 + \nabla \cdot \mathbf{K} \nabla T + \mu \nabla^2 \mathbf{v} \tag{1}
\]

Combining mass, momentum and Eq. (1), one can easily derive the entropy equation.

\[
\rho \left( \frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S \right) = \frac{K_T |\mathbf{V}|^2}{\Gamma^2} + \frac{\mu^2 \Phi}{T} + \nabla \cdot (\frac{K_T \nabla T}{T}) \tag{2}
\]

where \( S = s \ln \left( \frac{\rho}{\rho_0} \right) = \) entropy

\[
\Gamma = \frac{C_p}{C_v} \quad \text{ratio of specific heats}
\]
If all dissipative effects can be neglected \((k_r = 0, \mu > 0)\), then the entropy is conserved along the fluid element trajectories.

3. \(\partial_t S + \nabla \cdot \mathbf{v} S = \frac{DS}{Dt} = 0\) \(S = \text{constant}\) along fluid motion.

A simple case is the one where the entropy was originally uniform, \(S(x, t = 0) = S_0 = \text{constant}\). Eq. (3) then yields that \(S = S_0\) at all times.

\[
S(x, t) = S(x, 0) = S_0 \implies \frac{P(x, t)}{\rho(x, t)} = \frac{P(x, 0)}{\rho(x, 0)} = C^2
\]

We are now ready to study the stability of two compressible superimposed fluids. The fluids are assumed to be ideal gases with no dissipation, the governing equations are:

4. \(\partial_t p + \nabla \cdot (p \mathbf{v}) = 0 \quad \text{Momentum}\)

5. \(p \left(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \rho g
\]

For simplicity we assume \(\Gamma = 1\) (isothermal flow).
Equilibrium

\[ V_0 = 0 \]

\[ \frac{dP_0}{dz} = \rho_0 g \frac{dz}{dz} = -P_0 g \]

\[ P_0 = C_0^2 \] (for \( R = 1 \), \( C_0 \) is equal to the square of)

\[ \text{the sound speed and} \]

\[ \text{it can be different for the two fluids} \]

\[ C_0 = C_1, \; C_0 = C_2 \]

\[ \text{Substitute (9) into (8).} \]

\[ C_0^2 \frac{dP_0}{dz} = -P_0 g \rightarrow P_0 = P_0(z=0) e^{-\frac{gz}{C_0^2}} \]

In fluids 1 and 2, we find

\[ P_0 = P_1 e^{-\frac{gz}{C_1^2}} \quad Z > 0 \]

\( \text{(10)} \)

\[ P_0 = P_2 e^{-\frac{gz}{C_2^2}} \quad Z < 0 \]

\( \text{(11)} \)

\[ \begin{align*}
P_01 &= \beta_1 C_1^2 e^{-\frac{gz}{C_1^2}} \quad Z > 0 \\
P_02 &= \beta_2 C_2^2 e^{-\frac{gz}{C_2^2}} \quad Z < 0
\end{align*} \]

\( \text{Here} \)

\[ \beta_1 C_1^2 = \beta_2 C_2^2 \]

\[ \beta_1 \text{ and } \beta_2 \text{ are constants. Also define } L_1 = \frac{C_1^2}{g} \]

\[ L_2 = \frac{C_2^2}{g} : \text{stratification lengths.} \]
Stability

Linearized equations:

\[ \nabla \tilde{\varphi} + \nabla \cdot \rho + \rho \nabla \cdot \tilde{v} = 0. \quad (14) \]

\[ \rho \tilde{v} = -\nabla \tilde{\varphi} + \frac{\rho \tilde{g}}{\rho_0} \quad (15) \]

\[ \tilde{\rho} = \gamma \frac{c_o^2}{\rho_o} \tilde{\varphi}, \quad \tilde{\rho} = \gamma \frac{c_o^2}{\rho_o} \tilde{\rho} \quad (\gamma = 1) \quad (16) \]

Eq. (14):

\[ \tilde{\varphi} = \frac{x}{2} \frac{\rho_o e^{2x/l_o}}{L_o} - \frac{x}{2} \frac{\rho_o e^{2x/l_o}}{L_o} \]

Define, \( \tilde{M} \equiv \tilde{\varphi} \). Then

\[ \tilde{M} = \frac{x}{2} \frac{\rho_o e^{2x/l_o}}{L_o} \]

\[ \frac{x}{2} - \frac{x}{L_o} + \nabla \cdot \tilde{v} = 0 \quad (17) \]

Eq. (15)

\[ \gamma \tilde{v} = -\frac{c_o^2}{\rho_o e^{2x/l_o}} \nabla \tilde{\varphi} + \frac{\gamma}{\rho_o e^{2x/l_o}} \tilde{g} \]

\[ \gamma \tilde{v} = -\frac{c_o^2}{\rho_o e^{2x/l_o}} \left( \nabla \tilde{M} - \tilde{\varphi} \nabla \frac{e}{\rho_o} \right) + \frac{\gamma}{\rho_o} \tilde{g} \]
\[ \gamma \ddot{m} = -c_0^2 \left( \nabla \ddot{m} - \nabla \ddot{m} \frac{\hat{e}_z}{L_0} \right) + \dddot{m} \hat{g} \]  

Take \( \nabla \cdot (18) \rightarrow \gamma \nabla \cdot \ddot{m} = -c_0^2 \nabla^2 \ddot{m} \)

Take \( \dddot{m} \) (18) \( \rightarrow \gamma \dddot{m} = -c_0^2 \frac{\partial \ddot{m}}{\partial z} \)

Substitute into (17)

\[ \gamma^2 \ddot{m} + \frac{c_0^2}{L_0} \frac{\partial \ddot{m}}{\partial z} - c_0^2 \nabla^2 \ddot{m} = 0. \]

\[ \gamma^2 \ddot{m} + g \frac{\partial \ddot{m}}{\partial z} - c_0^2 \nabla^2 \ddot{m} = 0. \]

Take Fourier transform \( \rightarrow \partial_x \rightarrow iK \)

\[ \gamma^2 \ddot{m} + g \frac{\partial \ddot{m}}{\partial z} - c_0^2 \left( \partial_x^2 \ddot{m} - K^2 \ddot{m} \right) = 0. \]

Solution: \( \ddot{m} = e^{\alpha z} \)

\[ \gamma^2 - c_0^2 (\alpha^2 - K^2) + \alpha g = 0. \]

\[ \alpha^2 - \frac{g}{c_0^2} \left( \frac{\gamma^2}{c_0^2} + K^2 \right) = 0 \quad \alpha^2 - \frac{(g - \frac{K^2 c_0^2}{c_0^2})}{L_0} = 0 \]
\[
\alpha = \frac{1}{2L_0} \sqrt{1 + \frac{\frac{\alpha^2}{G}}{4C_0^2 + \frac{\delta^2 + K_C^2}{C_0^2}}}
\]

For simplicity define, \(\frac{\alpha^2}{G} = \frac{\delta^2}{K_C} \).

Then \( \frac{L^2}{C_0^2} = \frac{\delta^2 K_C}{C_0^2} = \frac{\delta^2 L}{L_0} \).

\[
\alpha = \frac{1}{2L_0} \left[ 1 + \sqrt{1 + 4 \left( \frac{\delta^2 K_C + K_{36}^2}{\delta^2 L} \right)^2} \right]
\]

Apply the boundary conditions:

\( \tilde{m}(z = \pm \infty) \rightarrow 0 \) and finds:

\( \tilde{m}(z > 0) \): \( \tilde{M}_4 = \tilde{M}_4 e^{\frac{\alpha_+}{\delta^2} z} \) \quad \text{Fluid 1} \quad (19)

\( \tilde{m}(z < 0) \): \( \tilde{M}_2 = \tilde{M}_2 e^{\frac{-\alpha_+}{\delta^2} z} \) \quad \text{Fluid 2} \quad (20)

\( \tilde{V}_{21} = -\frac{C_1}{\delta} \alpha^* \tilde{M}_1 = -\frac{\alpha_+^* L_1}{\delta^2} \tilde{M}_4 \) \quad \text{Fluid 1} \quad (21)

\( \tilde{V}_{22} = -\frac{C_1}{\delta} \alpha_+^* \tilde{M}_2 = -\frac{\alpha_+^* L_2}{\delta^2} \tilde{M}_2 \) \quad \text{Fluid 2} \quad (22)

Here \( \alpha_+^* = \frac{1}{2L_2} \left[ 1 - \sqrt{1 + 4 \left( \frac{\delta^2 K_{12} + K_{36}^2}{\delta^2 L_2} \right)^2} \right] \), \( \alpha_+ = \frac{1}{2L_2} \left[ 1 + \sqrt{1 + 4 \left( \frac{\delta^2 K_{12} + K_{36}^2}{\delta^2 L_2} \right)^2} \right] \).
Apply jump condition at the interface and use interface equation. Same as the incompressible case.

\[ \left[ V_2 \right]_{z=0} \rightarrow \text{interface linear equation} \quad (23) \]

\[ \left[ V_{21} = V_{22} \right]_{z=0} \rightarrow \text{no penetration} \quad (24) \]

\[ \left[ p_1 + \left( \frac{d P_{1z}}{d z} \right) \tilde{z} \right]_{z=0} = \left[ p_2 + \left( \frac{d P_{2z}}{d z} \right) \tilde{z} \right]_{z=0} \rightarrow \text{Newton's law} \quad (25) \]

From (24) and (21, 22) \[
\alpha_1 L_1 \mathbf{M}_1 = \alpha_2 L_2 \mathbf{M}_2
\]

From (23) and (21) \[
\tilde{\eta} = \frac{V_2}{\eta} = \frac{V_{21} - V_{22}}{\eta} = -\frac{\alpha_1 L_1 \mathbf{M}_1}{K \tilde{\eta}^2}
\]
Apply (25). We:

\[ \begin{align*}
\tilde P_1 &= \tilde P_1 \bigg|_{z=0} = \beta_1 C_1 \tilde M_1, \\
\tilde P_2 &= \tilde P_2 \bigg|_{z=0} = \beta_2 C_2 \tilde M_2
\end{align*} \]

\[ \begin{align*}
\frac{d\tilde P_1}{dz} &|_{z=0} = -\beta_1 g \\
\frac{d\tilde P_2}{dz} &|_{z=0} = -\beta_2 g
\end{align*} \]

Rewrite (25)

\[ \begin{align*}
\beta_1 C_1 \tilde M_1 - \beta_2 g \tilde H &= \beta_1 C_2 \tilde M_2 - \beta_2 g \tilde H \\
\downarrow
\end{align*} \]

\[ \begin{align*}
\beta_1 C_2 \tilde \eta_1 - \beta_2 (C_4) \tilde \eta_4 &= -\left(\beta_1 - \beta_2\right) g \frac{\alpha_1 \tilde L_1}{k^2} \tilde M_1 \\
\alpha_2 L_2
\end{align*} \]

\[ \left(\alpha_1 \tilde L_1 - \alpha_2 L_2\right) = \left(\beta_1 - \beta_2\right) g \frac{\alpha_1 \tilde L_1 \left(\alpha_1 \tilde L_1 - \alpha_2 L_2\right)}{\beta_1 C_1} < \frac{\tilde \eta_4}{k^2}. \]

\[ \left(\alpha_1 \tilde L_1 - \alpha_2 L_2\right) = \left(\frac{\beta_1}{\beta_1} \right) \left(\alpha_1 \tilde L_1 \right) \left(\alpha_2 L_2\right) < \frac{\tilde \eta_4}{k L_1}. \]

\textbf{Dispersion Relation}
Consider the short wavelength limit \((K \rightarrow \infty)\)

\[\alpha_1^+ \rightarrow \infty, \quad \alpha_2^+ \rightarrow \infty\]

**Dispersion Relation:**

\[+K\left(L_1 + L_2\right) = \frac{\rho_1 - \rho_2}{\rho_2} \left(\frac{\gamma^2}{\gamma_{12}}/L_1 + L_2\right)\]

\[
\gamma^2 = \frac{\rho_1 - \rho_2}{\rho_2} \frac{L_2}{L_1 + L_2} + \frac{\rho_1 - \rho_2}{\rho_2} \frac{1}{1 + \frac{c^2}{c^2}}
\]

\[
\gamma^2 = \frac{\rho_1 - \rho_2}{\rho_2} \frac{1}{1 + \frac{\rho_1 \beta^2 \rho_2}{\rho_1 \beta^2 \rho_2}} = \frac{\rho_1 - \rho_2}{\rho_1 \rho_2} A = \text{Atwood number}
\]

\[K \rightarrow \infty \quad \gamma^2 = A \Rightarrow \gamma = \sqrt{KgA}. \quad \text{(Remember that) \quad \gamma^2 = \frac{\delta^2}{\rho g}}\]

Short wavelength behaves as incompressible.
\( \theta_2 = 0 \) case! \( (\theta_{02} \text{ stays finite and } c_2 \to \infty) \).

Then \( l_2 \to \infty \).

Dispersion relation:

\[ -\theta_2/l_2 = \frac{\alpha^{-1} l_1 \theta_2}{l_2} \Rightarrow \frac{\theta_2}{l_2} = -\frac{\alpha^{-1} l_1}{K} \]

\[ \frac{\theta_2}{l_2} = -\frac{1}{2 Kl_1} \left\{ 1 - \sqrt{1 + 4\left(\frac{\alpha^{-2} l_1^2}{K l_1} + \frac{K^2 l_1^2}{K^2 l_1^2}\right)} \right\} \]

\[ \frac{\alpha^{-4} K^2 l_1^2 + 4 K l_1^2 \theta_2^2 + 1}{\alpha^{-4} K^2 l_1^2} = 1 + 4 \frac{\theta_2^2}{K l_1} + \frac{K^2 l_1^2}{K^2 l_1^2} \]

\[ \frac{\theta_2^4}{K} = 1 \Rightarrow \gamma = \sqrt{K q} \]

If the Atwood number \( \to 1 \), then the instability behaves as incompressible.
Long wavelength modes: $KL_1 \ll 1$, $KL_2 \ll 1$, $\gamma^2 \ll 1$

$$\alpha^2 \approx \frac{1}{\gamma^2 KL_1} \left\{ \gamma - \gamma^2 - \gamma^2 KL_1 - \gamma^2 KL_2 \right\} = -\frac{\left(\gamma^2 + KL_1\right)KL_1}{L_1}.$$

$$\alpha^2 \approx \frac{1}{L_2}.$$

Dispersion relation for $KL_2 \ll 1$.

- $(\gamma^2 + KL_1)KL_2 - 1 = \left(1 - \frac{p_2}{p_1}\right) - \left(\frac{\gamma^2 + KL_1}{\gamma^2 KL_1}\right) KL_2$

$$\gamma = \gamma - \frac{p_2}{p_1} + \frac{KL_1}{\delta^2} \left(1 - \frac{p_2}{p_1}\right)$$

$$\frac{\gamma^2}{KL_1} = \left(\frac{\delta}{p_2}\right) \left(1 - \frac{p_2}{p_1}\right) \Rightarrow \gamma^2 = K g KL_1 \left(\frac{\delta}{p_2} - 1\right)$$

$$\gamma = K V g L_1 \left(\frac{\delta}{p_2} - 1\right) = K C_1 \sqrt{\frac{\delta}{p_2} - 1} = K \sqrt{C_e^2 - C_l^2}$$

sound speed.

Compare with incompressible growth rate:

$$\gamma_{inc} = \sqrt{AKg} \quad A = \frac{p_1 - p_2}{p_1 + p_2}.$$
Rewrite the compressible growth rate for $KL << 1$,

$$\gamma = \sqrt{AKg} \sqrt{KL_1} \sqrt{\frac{\rho_1 + 1}{\rho_2}} = \text{time} \sqrt{KL_1} \sqrt{1 + \frac{\rho_1}{\rho_2}}$$

Since $KL_1 << 1$, then $\gamma \ll \text{time}$ if $\rho_1/\rho_2$ is finite.

We have shown before that if $\rho_2 \to 0$ then

$\gamma$ approaches the incompressible value $\gamma \to \sqrt{AKg}$.

In conclusion: Long wavelength modes in a compressible medium grow slower than in an incompressible one. This conclusion is misleading because the fluid masses are finite for compressible and end infinite for incompressible. One should compare the compressible case with a finite thickness incompressible case, with equal mass.

![Typical spectrum graph](image-url)
The effect of surface tension.

If surface tension is present at the fluid interface, we expect the instability growth rate to be reduced as extra energy must be spent by the instability to bend the interface.

Surface tension is introduced as a force acting on the interface.

Here $T$ is the tension and $R$ is the surface radius of curvature. The pressure balance at the interface requires that:

$$(p_2 - p_1) \, R \, d\theta = T \, d\theta$$
Linearize the pressure balance and remember that:

\[ \frac{1}{R} = \left( \frac{1}{R_0} \right) + \left( \frac{1}{R_1} \right) = \frac{1}{R_0} + \frac{1}{\sqrt{2g \theta_x \tilde{\gamma}}} \left( 1 + \left( \frac{\delta \theta}{\theta_0} \right)^2 \right)^{3/2} \]

thus leading to:

\[ \frac{\partial P_1}{\partial z} + \frac{dP_1}{dz} \tilde{\gamma} + \tilde{\gamma}_z - \frac{dP_2}{dz} \tilde{\gamma} - \tilde{\gamma}_z = T \theta_x \tilde{\gamma} \left( \frac{\partial \theta_x}{\partial \gamma} \right)^2 = -K^2 \tilde{T} \tilde{\gamma} \]

Use equilibrium relation:

\[ \frac{dP_1}{dz} = -P_0 \tilde{\gamma}_z \text{ (here } g > 0) \]

and finally the jump condition:

\[ \tilde{P}_1 - \tilde{P}_2 = (P_0_1 - P_0_2) \tilde{\gamma}_z \tilde{g} = -K^2 \tilde{T} \tilde{\gamma} \]

Let's go back to the incompressible relations:

1. \( \tilde{\gamma}_z = \tilde{\gamma}_z \)
2. \( \tilde{\gamma}_z = \tilde{\gamma}_z \)
3. \( P_1 - P_2 = (P_0_1 - P_0_2) \tilde{\gamma}_z = K^2 \tilde{T} \)
4. \( \tilde{\gamma}_z = K \tilde{\gamma} e^{-Kz}, \quad \tilde{\gamma}_z = -K \tilde{\gamma} e^{-Kz} \)
5. \( \tilde{P}_1 = P_0_1 \tilde{\gamma} e^{-Kz}, \quad \tilde{P}_2 = P_0_2 \tilde{\gamma} e^{Kz} \)
Substitute (4) and (5) into (1,2,3) and set the determinant to zero to find the dispersion relation:

\[ \chi^2 = \left( \frac{(\rho_{01} - \rho_{02})g}{\rho_{01} + \rho_{02}} \right) K \]

\[ \chi = \sqrt{AKg - \left( \frac{K^2 T}{\rho_{01} + \rho_{02}} \right)} \]  

growth rate without surface tension.

Eq. (6) shows that surface tension is stabilizing (as expected). Furthermore, Eq. (6) shows that perturbations with wave number \( K \) greater than:

\[ K^2 > \left( \frac{\rho_{01} - \rho_{02})g}{T} \right) \]

are stable. The wave number

\[ K_c = \sqrt{\left( \frac{\rho_{01} - \rho_{02})g}{T} \right)} \]

is called the cutoff wave number.

Typical dispersion curve
Incompressible Rayleigh-Taylor instability of a finite thickness layer

Consider the heavy layer of fluid supported by a lighter fluid.

\( p = p_3 = 0 \) free surface.

\[
\begin{array}{c}
\text{heavy} & p = p_4 \uparrow d \\
\text{light} & p_2 \downarrow g
\end{array}
\]

Equilibrium.

**Region 1:** \[ \frac{dP_4}{dz} = -p_2 g \quad P_1 = -P_1 g z + c \]

**Region 2:** \[ \frac{dP_2}{dz} = -p_2 g \quad P_2 = -p_2 g z + c \]
Linear Stability.

Region 1: Some equations as for the infinite medium:

\[ \tilde{V}_4 = -\nabla \tilde{\Phi}_4, \quad \tilde{\Phi}_4 = \frac{\tilde{\Pi}_4}{\rho_1}, \quad \text{where} \]

\[ \frac{\partial^2 \tilde{\Phi}_4}{\partial z^2} - k^2 \tilde{\Phi}_4 = 0 \]

\[ \tilde{\Phi}_4 = \tilde{A}_4 e^z + \tilde{B}_4 e^{-z} \]

We cannot set \( A_1 = 0 \) because the medium 1 is finite. Derive \( \tilde{\nabla}_z^2 = -\partial_z^2 \tilde{\Phi}_4 \) and \( \tilde{\Pi}_1 \)

\[
\begin{cases}
\tilde{V}_4 = -k \left( \tilde{A}_4 e^z - \tilde{B}_4 e^{-z} \right), \quad (1) \\
\tilde{\Pi}_1 = g_4 \tilde{\Phi}_4 = g_4 z \left[ \tilde{A}_4 e^z + \tilde{B}_4 e^{-z} \right]. \quad (2)
\end{cases}
\]

Apply the boundary condition at the free surface

\[ z = \bar{s}(x,t) + \delta \]

\[ z = \eta(x,t) \]

\[ z = \delta \]

\[ z = \bar{s}(x,t) \]

\[ z = \delta \]

\[ z = \bar{s}(x,t) + \delta \]

\[ z = \eta(x,t) \]

\[ z = \delta \]

\[ z = \bar{s}(x,t) \]

\[ z = \delta \]
Definition of free surface:

\[ p = \text{constant on the surface}. \]

Thus \( p(z = d + \frac{2}{3}, x, y) = p_0 = \text{constant}. \)

\[ \frac{p_0}{
\begin{align*}
&+ \left( \frac{dp_0}{dz} \right) \bigg|_{z = d} \left( \frac{2}{3} \right)^2 + \\
&+ \frac{2}{3} \bigg|_{z = d} = p_0 = > \right. \\
&\left. \right. \\
&- f_1g 
\end{align*}
\]

Also: the surface moves with the fluid \( 1, \)

\[ \frac{\partial \zeta}{\partial t} = - V_x \frac{\partial \zeta}{\partial x} + V_y \bigg|_{z = d} \]

\[ \frac{\partial \zeta}{\partial z} = V_z \bigg|_{z = d} \]

Substitute 1 and 2 into 3 and 4.

\[ \beta_k [A_1 e^{kd} + B_1 e^{-kd}] = \beta g \frac{\zeta}{\zeta} \]

\[ \zeta = -\frac{k}{\beta} \left( A_1 e^{kd} - B_1 e^{-kd} \right) \]
Combine the two equations:

\[ A_1 e^{+} + B_1 e^{-} = -\frac{K\theta}{\delta^2} \left( A_1 e^{+} - B_1 e^{-} \right) \]

\[ \frac{\tilde{A}_1}{B_1} = \frac{-2K\delta}{e^{-}\left( \frac{K\theta}{\delta^2} - 1 \right)} \]

\[ \alpha = \frac{K\theta/\delta^2 - 1}{K\theta/\delta^2 + 1} \]

It follows that

\[ \tilde{V}_{1z} = KB_1 \left[ e^{-\alpha e} \right] \]

\[ \tilde{P}_{1z} = P_z \delta B_1 \left[ e^{+\alpha e} \right] \]

**Fluid 1**

**Fluid 2** is semi-infinite \(\rightarrow\) solution as in previous lecture.

Apply jump conditions at fluid 1-2 interface:

\[ (\tilde{V}_{1z} = \tilde{V}_{2z})_{z=0} \quad (\tilde{P}_{1z} = \tilde{P}_{2z})_{z=0} \]

\[ (\tilde{P}_1 - \tilde{P}_2) = (\tilde{P}_1 - \tilde{P}_2) \delta \tilde{V} \]

\[ \tilde{V}_{1z} = 5 \quad \tilde{V}_{2z} = -K\tilde{A}_2 e^{+Kz} \]

\[ \tilde{P}_1 = 6 \quad \tilde{P}_{2z} = P_2 \delta \tilde{A}_2 e^{-Kz} \]
Substitute.

\[ K\tilde{B}_1 \left[ 1 - \alpha e^{-2Kd} \right] = -K\tilde{A}_2 = 0 \]  \( \blacksquare \)

\[ f_1 \tilde{B}_1 \left[ 1 + \alpha e^{-2Kd} \right] - f_2 \tilde{A}_2 = (f_1 - f_2) gZ \]  \( \blacksquare \)

\[ f_1 \tilde{B}_1 \left[ 1 + \alpha e^{-2Kd} \right] = \left[ f_2 \gamma - \frac{(f_1 - f_2) KG}{f_2} \right] \tilde{A}_2 \]  \( \blacksquare \)

Divide (8) by (7)

\[ f_1 \frac{1 + \alpha e^{-2Kd}}{1 - \alpha e^{-2Kd}} = f_2 \frac{1 - \left( \frac{f_1 - f_2}{f_2} \right) KG}{f_2^2} \]

\[
\frac{f_2}{f_2} \frac{1 + \alpha e^{-2Kd}}{1 - \alpha e^{-2Kd}} + 1 - \left( \frac{f_2}{f_2} - 1 \right) \frac{KG}{f_2} = 0.
\]

\[
\text{Dispersion relation}
\]

where \[ \alpha = \frac{KG/f_2 - 1}{KG/f_2 + 1}. \]

Solve the dispersion relation to find \( \gamma \).
Growth Rate

\[ \gamma = \sqrt{AKg \frac{1 - e^{-2Kd}}{1 - A e^{-2Kd}}}, \quad \text{growth rate.} \]

Observe: if \( A \to 1 \) then \( \gamma = \sqrt{AKg} \) as the semi-infinite medium result.

If \( Kd \gg 1 \) then \( \gamma \to \sqrt{AKg} \) as the semi-infinite medium result.

If \( Kd \ll 1 \), then

\[ \gamma = \sqrt{AKg} \cdot \sqrt{\frac{2Kd}{1 - A}}, \]

rewrite \( \frac{2}{1 - A} = \frac{2}{1 - \gamma^2} = \frac{2(r+1)}{\gamma^2} = (r+1) \)

and compare

\[ \gamma_{\text{finite layer}} = \sqrt{AKg} \sqrt{Kd} \sqrt{1 + \frac{f}{f_2}}, \quad \text{for } Kd \ll 1 \]

\[ \gamma_{\text{compressible}} = \sqrt{AKg} \sqrt{KL_2} \sqrt{\frac{f_2}{f}}, \quad \text{for } KL \ll 1 \]

Notice the similarity?


APPENDIX
Manipulations of the dispersion relation (optional).

Write \( \frac{k_g}{f_2} = \frac{-1 - \alpha}{\alpha - 1} = \frac{1 + \alpha}{1 - \alpha} \).

\[ \frac{p_4}{f_2} \frac{1 + \alpha e^{-2K_d}}{1 - \alpha e^{-2K_d}} + 1 + \frac{1 + \alpha}{1 - \alpha} - \frac{p_4}{f_2} \frac{1 + \alpha}{1 - \alpha} = 0 \]

\[ \frac{p_4}{f_2} \left( \frac{1 + \alpha e^{-2K_d}}{1 - \alpha e^{-2K_d}} - 1 \right) + \frac{2}{1 - \alpha} = 0 \]

\[ \frac{p_1}{f_2} \frac{\alpha (e^{-2K_d} - 1)}{1 - \alpha e^{-2K_d}} + 1 = 0 \]

\[ \frac{p_4}{f_2} \alpha (e^{-2K_d} - 1) + 1 - \alpha e^{-2K_d} = 0 \]

\[ \alpha \left[ e^{-2K_d} \left( \frac{p_4}{f_2} - 1 \right) - \frac{p_4}{f_2} \right] = -1 \]

Define \( r = \frac{p_1}{p_2} \Rightarrow \alpha = \frac{1}{r - (r-1)e^{-2K_d}} \)

\[ r = \frac{\delta^2}{k_g} = \frac{1 - \alpha}{\alpha + 1} = \frac{k_{-1}}{k_{-1} + 1} = \frac{r - (r-1)e^{-2K_d}}{r - (r-1)e^{-2K_d} + 1} \]

\[ \frac{\delta^2}{k_g} = \frac{r - 1}{r + 1} \left[ 1 - e^{-2K_d} \right] = A \left[ 1 - e^{-2K_d} \right] \]

\[ \frac{\delta^2}{k_g} = r - 1 \left[ 1 - \frac{e^{-2K_d}}{r + 1} \right] = A \left[ 1 - \frac{e^{-2K_d}}{r + 1} \right] \]