Short wavelength R-T instability
The effect of finite heat conduction.

In the presence of finite heat conduction, the conservation equations can be written in the following form:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{mass} \]
\[ \rho (\mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \rho \mathbf{g} \quad \text{momentum} \]
\[ \rho c_v (\mathbf{v} \cdot \nabla T + \nabla \cdot \mathbf{v} T) = -p \nabla \cdot \mathbf{v} + \nabla \mathbf{K} \cdot \mathbf{v} \quad \text{Temp. Equation} \]
\[ p = \rho R T \quad \text{Equation of state} \]

Here \( K \) is the coefficient of thermal conductivity.

Equilibrium

\[ \frac{dp}{dz} = \rho g \frac{dz}{dz} = -\rho g(z) \frac{dz}{dz} \Rightarrow p = p_0 - \int_0^z \rho g(z) \frac{dz}{dz} \]

\[ \frac{d}{dz} [K \frac{dT}{dz}] = 0 \quad T = T_0 - q_0 \int_0^z \frac{dz}{K} \quad \text{(integration constant)} \]
Second: Use the WKB ansatz to study short wavelength stability.

\[ p^2 = -\nabla^2 \psi + \psi^2 \]

First: Linearize the equations of motion.

\[ \rho \psi'' + \psi' + \rho \psi = 0. \]

We study the stability of short wavelength model. This requires a \( k \) of this kind: \( k_L \) (not physical).
where $S$ and $A$ vary on the length scale and $K \to \infty$.

Substitute Ansatz into the linearized equations. Keep only the largest terms in $K$.

\[ \hat{\rho} + \hat{V}_2 \hat{p}' + \hat{p} \left( KS'V_2 + iK\hat{V}_x \right) = 0 \quad (5) \]
\[ \hat{p} \hat{V}_2 = -KS'\hat{p} - \hat{p}g \quad (6) \]
\[ \hat{p} \hat{V}_x = -iK\hat{p} \quad (7) \]
\[ \hat{p} \hat{V}_x = -\hat{p} \left( KS'V_2 + iK\hat{V}_x \right) + K\left(KS'^2 - K^2\right)\hat{V}_x \quad (8) \]
\[ \hat{p} = \hat{p}RT + \hat{p}RT \quad (9) \]

From (5):
\[ KS'V_2 + iK\hat{V}_x \approx 0 \quad \hat{V}_x = iS'\hat{V}_2 \quad (10) \]

From (7):
\[ \hat{p} = +\frac{\hat{p}K\hat{V}_x}{K} = -\frac{\hat{p}K\hat{V}_x}{K} \quad (11) \]

From (6):
\[ \hat{p} \hat{V}_2 = +KS'\hat{p} \left( S'\hat{V}_2 - \hat{p}g \right) \]
\[ \hat{p} = \left( S^2 - 1 \right) \hat{V}_2 = \hat{p}g \quad (12) \]
From (12) \( \hat{V}_z = \frac{\hat{P} \mathcal{L}}{\mathcal{P}(S^2-1)} \).

Substitute (10), (6) and (12) into (9).

\[ \rho CV \left[ \gamma \hat{\mathbf{T}} + \hat{V}_z \hat{\mathbf{T}}' \right] = \rho \left[ \gamma \frac{\hat{P} + \hat{V}_z \rho'}{\rho} \right] + KK^2 (S^2-1) \hat{T} \]  \( \text{(13)} \)

Use (9) and (11)

\[ \rho \hat{\mathbf{T}} + \hat{\rho} \mathbf{T} = -\frac{\rho \xi S \hat{V}_z}{K} \quad \text{small because of } \frac{\rho}{K} \]

\[ \rho \hat{\mathbf{T}} - \hat{\rho} \mathbf{T} \] \( \text{(14)} \)

Use (14), (12) into (13).

\[ \rho CV \left[ -\gamma \frac{\hat{P}}{\rho} + \frac{\hat{P}}{\rho} \frac{\xi S}{(S^2-1)} \right] = \rho \hat{\mathbf{T}} \left[ \frac{\hat{P}}{\rho} + \frac{\hat{P}}{\rho} \frac{\xi S}{(S^2-1)} \frac{\rho'}{\rho} \right] + 
\]

\[ + KK^2 (S^2-1) \left( -\frac{\hat{P}}{\rho} \mathbf{T} \right) \] \( \text{(14)} \)
Define \( L_p = \left( \frac{1}{\rho} \frac{d\rho}{dz} \right)^{-1} \), \( \hat{m} = \frac{\rho}{\rho_c} \), \( L_T = \left( \frac{1}{T} \right)^{-1} \).

Use \( c_v = R / (\Gamma - 1) \) or \( R = (\Gamma - 1) c_v. \)

\[
\hat{m} \rho c_v T \left\{ \frac{1}{\gamma - 1} - \frac{\gamma}{L_T \gamma (S^2 - 1)} \right\} = (\Gamma - 1) \rho c_v T \left\{ \frac{1}{\gamma - 1} - \frac{\gamma}{L_T \gamma (S^2 - 1)} \right\} \frac{\hat{m} T}{\rho c_v T}
\]

\[
= \frac{\kappa k^2 (S^2 - 1)}{\rho c_v T} \left\{ \frac{1}{\gamma - 1} - \frac{\gamma}{L_T \gamma (S^2 - 1)} \right\} \frac{\hat{m} T}{\rho c_v T}
\]

\[
\frac{\kappa k^2 (S^2 - 1)^2}{\rho c_v T} = (\Gamma - 1) \left[ (S^2 - 1) + \frac{8/4T}{\delta^2} \right] + \left[ (S^2 - 1) - \frac{8/4T}{\delta^2} \right]
\]

\[\rho c_v T = \frac{P}{\Gamma - 1} \text{ constant} \quad \hat{L}_p \approx - L_T
\]

\[
\frac{\kappa k^2}{\gamma \rho c_v T} (S^2 - 1)^2 = \Gamma \left[ (S^2 - 1) + \frac{8/4T}{\delta^2} \right]
\]
\[(S^2 - 1)^2 \frac{d^2}{d\sigma^2} \alpha = (S^2 - 1) - \frac{9/l_0}{\delta^2} = 0\]

\[\alpha = \frac{KK^2}{YPC_P} \quad C_\alpha = \Gamma_CV\]

Solve for \(S^1\):

\[S^1 = 1 \pm \frac{1}{2\alpha} \sqrt{1 + 4\alpha \frac{9/l_P}{\delta^2}}\]

In order to satisfy the boundary conditions, the perturbation must vanish at \(z \to \pm \infty\). Therefore we can only use \(S^1^+\) for \(z \to -\infty\) and \(S^1^-\) for \(z \to +\infty\).
It is necessary that the $S^+$ root changes into the $S^-$ at some point. This can only occur if \( Q = 1 + 2\alpha \pm \sqrt{1 + 4\alpha^2 / \ell^2} \) becomes zero at some point.

Observe that the $Q^+$ can never be zero because $\alpha > 0$ and $g/L_\alpha > 0$.

Therefore, only $Q$ can be retained and the minimum of $Q$ must be zero for the solution $S^+$ to change from $S^+$ to $S^-$.

\[ Q_{\text{min}} = 0 \Rightarrow 1 + 2\alpha_m = \sqrt{1 + 4\alpha_m g / L_\alpha / \ell^2} \]

where $\alpha_m$ and $L_\alpha$ are calculated at the point where $Q^-$ is minimum.
It follows that

\[ x + \frac{4m^2}{\lambda} \frac{4m^2}{x} = x + \frac{4m}{\lambda} \frac{9/Lm}{x^2} \]

\[ \gamma^2 + \gamma \frac{2mK^2}{PmC_p} - \frac{9}{Lm} = 0 \]

\[ \gamma^2 + \frac{K_mK^2}{P_mC_p} \gamma - \frac{9}{Lm} = 0 \]

Only the + root is unstable.

\[ \gamma = -\frac{K_mK^2}{2P_mC_p} + 4\sqrt{\frac{9}{Lm} + \left(\frac{K_mK^2}{2P_mC_p}\right)^2} \]

For \( K \to 0 \Rightarrow \gamma \approx \frac{9}{Lm} \Rightarrow \text{no effect of thermal conduction} \)

For \( K \to \infty \Rightarrow \gamma \approx \frac{9}{Lm} \frac{P_mC_p}{K_mK^2} \Rightarrow 0 \)

Thermal conduction is stabilizing for short wavelengths.
Rayleigh-Taylor instability of accelerated ablation fronts.

The effect of thermal conduction

Ablation fronts in laser accelerated targets.

Ablation front moving inside the target.
We call "ablation velocity \( \mathbf{V}_a \)" the velocity at which the front moves inside the target.

The target is accelerated in the direction of the laser with acceleration \( \mathbf{a} \).
If $P_a$ is the pressure applied by the laser onto the target surface and there is vacuum on the other side of the target, then:

$$\rho g = S (P_a - 0)$$

Target mass = $\rho S d$

Denote with $V_0$ the velocity of the ablation front inside the target. Then, in the ablation front frame of reference, we have the following configuration:

$U_{\text{heavy}} = -V_{\text{ablation}}$

$U_{\text{light}} = -V_{\text{blow-off}}$

Diagram:

- Solid target (Heavy)
- Ablated material (Light)
- Ablation front
- Blow-off velocity
- Z = 0
This is an unstable configuration because similar to the heavy fluid supported by a light fluid with gravity directed from heavy to light.

Equilibrium solutions:

Expected profiles:

\[ p \quad \text{pTarget} \quad \text{TTarget} \quad U \quad \text{UIC: velocity} \quad g. \]

\[ U_{\text{IC}} \]

\[ \delta \]

\[ \delta = \text{ablation front width} \ll d = \text{target thickness}. \]

Equilibrium equations:

\[ \nabla \cdot \mathbf{p} U = 0 \quad \rho U = S_{\text{IC}} U_{\text{IC}} = \text{const}. \]

\[ \nabla \cdot (\rho + \rho U^2) = -\rho g \]

Energy

\[ \nabla \cdot \left[ \left( \frac{\rho p}{\Gamma - 1} + \frac{\rho U^2}{2} \right) U - K_2 \nabla T \right] = \rho g U = S_{\text{IC}} U_{\text{IC}} g \]

\[ \left( \frac{\rho p}{\Gamma - 1} + \frac{\rho U^2}{2} \right) U - K_2 \nabla T = \left( \frac{\rho p}{\Gamma - 1} + \rho \frac{U^2}{2} \right) U - K_2 \frac{S_{\text{IC}} U_{\text{IC}} g}{\rho_{\text{IC}} U_{\text{IC}} g^2}. \]
\( k = \text{thermal conductivity} = k_T \left( \frac{I}{I_T} \right)^\nu, \quad \nu = \frac{5}{2} \)

\( T = \text{temperature} \)

\( p < p_R T \leftarrow \text{equation of state} \)

Assume \( \text{s}ubl\text{sonic} \text{ flow} \quad p u^2 \ll p \)

\[
\text{Momentum} \quad \frac{dP}{dz} = -pg \\
\frac{1}{P} \frac{dP}{dz} = \frac{1}{P} = -\frac{pg}{P} \\
\]

Remember \( \frac{g}{P_n} \rightarrow \frac{1}{P} \rightarrow \frac{1}{L_p} \rightarrow L_p \approx \frac{1}{\partial \ell} \)

Here \( L_p \) is the pressure gradient scale length.

\( L_p \approx \ell \) implies that the pressure change on the scale \( \ell \).

Since \( p, u \) are expected to change over a scale length \( \delta \ll \ell \), we can treat the pressure as a constant over the scale length \( \delta \).
Rewrite energy eq. using \( p \frac{U^2}{c^2 < p, K_0 \gg 1} \) and \( p \) constant and:

\[
\begin{align*}
\frac{\rho U_h}{\rho_T} \frac{\Gamma P}{\Gamma - 1} - K_T \left( \frac{\Gamma}{\rho_T} \right)^\frac{3}{2} & = U_h \frac{\Gamma P}{\Gamma - 1} - \frac{K_T}{\rho_T} + \rho_g U_h \bar{z} \\
\bar{T} & = \frac{P}{\rho R} \\
\frac{\rho U_h}{\rho_T} - K_T \left( \frac{\Gamma}{\rho_T} \right)^\frac{3}{2} & = U_h. \\
\frac{K_T}{\rho_T} \frac{\Gamma - 1}{R \rho_T} \left( \frac{\rho}{\rho_T} \right) + U_h \left( 1 - \frac{\rho}{\rho_T} \right) & = 0
\end{align*}
\]

\[
- \frac{\partial}{\partial z} \left( \frac{1}{3} \right) + \left( 1 - \frac{1}{3} \right) \bar{s}' = 0
\]

\[
\frac{d}{ds} = (1 - s) \bar{s}'
\]

\[
\frac{\bar{s}}{s} = \frac{p}{p_T}
\]

\[
\frac{\bar{z}}{z} = \left( \frac{z}{L_0} \right)
\]

Observe \( L_0 > 0 \) because \( U_h > 0 \).

\( L_0 \) is the typical front thickness.
Simple case: \( V = 0 \rightarrow \) constant heat conduction.

\[
\frac{1}{\frac{\partial S}{\partial z}} = -1 \quad \ln \left( \frac{S}{S-1} \right) = \frac{1}{2} z + c_0
\]

\[
S - 1 = c_1 e^{\frac{1}{2} z} \quad 3 = 1 + c_1 e^{\frac{z}{2}} = \frac{3}{\rho}
\]

\[
\rho = \frac{\rho_0}{1 + c_1 e^{\frac{z}{2}}} = \frac{1}{1 + e^{\frac{z}{2} - \frac{z_0}{2}}} \quad \text{where} \quad e = c_1
\]

\( z_0 \) is the typical thickness of the ablation front.

Since \( \rho U = \) constant, we can immediately plot the velocity profile (\( U \propto 1/\rho \))
General power law for heat conduction

\[ R = R_0 \left( \frac{T}{T_0} \right)^n \]

\[ \frac{dS}{dz} + S^{\nu+1} (S-1) = 0 \quad \text{--- density equation} \]

1) Solve near the target where \( S \approx 1 \) and \( \hat{z} \gg 1 \)

\[ \frac{dS}{dz} \approx (1-S) \implies \theta u (1-S) = -\frac{\hat{z}}{\nu} + C_0. \]

\[ 1 - S = e^{-(\hat{z} - C_0)} \quad \Rightarrow \quad S = 1 - e^{- (\hat{z} - C_0)} \]

Small correction depending on the origin of the reference frame.

2) Solve in the blow-off region where \( S \ll 1 \) and \( |\hat{z}| \gg 1 \)

\[ \frac{dS}{dz} + S^{\nu+1} (S-1) \approx 0. \]

\[ \frac{dS}{dz} = -S^{\nu+1} \quad \frac{1}{dS} = -S^n \frac{dS}{dz} \]

\[ -\frac{1}{v} \frac{d}{dz} \left( \frac{1}{S^n} \right) = 1 \quad \frac{1}{S^n} \approx -\nu \frac{dS}{dz} \quad S = \left( \frac{-1}{\nu} \right)^{1/\nu} \quad (\frac{\nu}{|\hat{z}|})^{1/\nu} \]

Since \( S = \frac{\rho}{\rho_T} \rightarrow \text{the density decreases as} \ (\frac{\rho}{\rho_T})^{1/\nu} \)