In this notebook, we construct graphs of the amplitude response for sinusoidally forced oscillators. The basic differential equation is

\[
m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + c x = F_0 \cos(\gamma t) .
\]

As we showed in class, this equation has a general solution of the form

\[x(t) = x_T(t) + x_P(t),\]

where \(x_T\) is the transient part of the solution which decays exponentially with time, and \(x_P\) is the periodic particular solution which remains after the transients have died away. In class, we found the following formula for \(x_P\):

\[x_P(t) = A_P \sin(\gamma t + \theta),\]

where

\[A_P = \frac{F_0}{\left[(k - m \gamma^2)^2 + \gamma^2 b^2 \right]^{1/2}},\]

and

\[
\sin(\theta) = \frac{k - m \gamma^2}{\left[(k - m \gamma^2)^2 + \gamma^2 b^2 \right]^{1/2}},
\]

\[
\cos(\theta) = \frac{\gamma b}{\left[(k - m \gamma^2)^2 + \gamma^2 b^2 \right]^{1/2}}.
\]

The quantity of greatest interest here is the amplitude response \(A_P\). In this notebook we study how \(A_P\) depends on the driver frequency \(\gamma\) and the damping parameter \(b\). We begin by defining \(A_P\) for Mathematica.

\[In[22] := \text{Clear}[b, \gamma];\]

\[In[23] := A_P = \frac{F_0}{\text{Sqrt}\left[(k - m \gamma^2)^2 + \gamma^2 b^2 \right]}\]

\[Out[23] = \frac{\sqrt{b^2 \gamma^2 + (k - m \gamma^2)^2}}{F_0}\]
Our approach will be the following: (1) we choose particular values for the mass $m$, the spring constant $k$, and the force amplitude $F_0$, which remain fixed throughout the notebook; (2) we choose a set of values for $b$, and (3) for each value we plot $A_P$ as a function of the driver frequency $\gamma$. As we showed in class, this plot, called the amplitude response curve, has a maximum at a particular frequency, provided that $b^2 < 2km$. We can also express this condition in terms of the damping parameter $\zeta$, defined by

$$\frac{b}{2\sqrt{km}}.$$

In terms of $\zeta$, the condition for a maximum to occur is $\zeta^2 < 1/2$. Such a maximum is called a resonance, and the frequency at which the resonance occurs is called the resonant frequency. We denote the resonant frequency by $\gamma_r$. As we showed in class,

$$\gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}.$$

We are going to choose $m = 5$ kg, $k = 500$ N/m, and $F_0 = 500$ N. The damping parameter $\zeta$ is then given by $\zeta = b/(2\sqrt{km}) = b/100$. Now we define a function amp which is equal to $A_P$ with these fixed values of $m$, $k$, and $F_0$.

```
In[24]:= amp[b_, \[Gamma]_] = \[Alpha] /.(m \to 5, k \to 500, F0 \to 500)
```

```
Out[24]= 500
```

```
In[25]:= amp[50, \[Gamma]]
```

```
Out[25]= 500
```

We will be systematic later. For now, let's just look at three cases: a lightly damped system, a critically damped system and a heavily overdamped system. The classification is most easily done on the basis of the value of $\zeta$ which is equal to $b/100$ in the present case. For the lightly damped system, we will take $b = 10$, giving $\zeta = 0.1$. We then plot the amplitude response as a function of the driver frequency $\gamma$. We first define a function which will produce a graph for any given value of $b$, and which labels the graph with the value of $\zeta$.

```
In[26]:= ampres[b_] := Plot[amp[b, \[Gamma]], {\[Gamma], 0, 30}, AxesLabel -> {"\[Gamma]", "A_P"},
PlotLabel -> SequenceForm["Amplitude Response for $\zeta =$ ", N[b/100]], ImageSize -> 400]
```

First the lightly damped case (underdamped):
This graph has all of the important basics associated with the concept of resonance. We go over some of the features in the graph. For \( \gamma = 0 \), the amplitude response is the static deflection \( F_0/k \), which in this case is \( 500/500 = 1 \text{ m} \). As we increase the driver frequency, the amplitude response increases. It is sharply peaked at a frequency close to 10, which turns out to be the natural frequency of this system in the absence of damping:

\[
\text{Out}[28]=10
\]

The exact resonant frequency is

\[
\text{In}[29]:= N[Sqrt[k/m - (b^2/(2m^2))]] /. \{k \to 500, m \to 5, b \to 10\}
\]

\[
\text{Out}[29]=9.89949
\]

which is very close to 10. Our result is generally true: in a lightly damped system, the resonant frequency for driven motion is approximately equal to the natural frequency of the undamped system. Finally notice that the peak is rather narrow. The response is about 5 times larger at the resonance frequency than it is for frequencies of say 5 or 15 s\(^{-1}\). In this sense, the lightly damped system is "tuned" to the resonant frequency.

Now we look at a critically damped system. This corresponds to \( \zeta = 1 \), which we get by taking \( b = 100 \).
A strikingly different result. There is no peak at all now. The response simply decreases as the driver frequency increases. There is not even a hint of a resonance. For $\zeta = 1$, the undriven system does not oscillate -- the damping is just too great. When the system is driven, it simply follows in a kind of passive, sluggish way.

Now let's look at a heavily overdamped case. We take $\zeta = 5$, corresponding to $b = 500$. 

A strikingly different result. There is no peak at all now. The response simply decreases as the driver frequency increases. There is not even a hint of a resonance. For $\zeta = 1$, the undriven system does not oscillate -- the damping is just too great. When the system is driven, it simply follows in a kind of passive, sluggish way.

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A strikingly different result. There is no peak at all now. The response simply decreases as the driver frequency increases. There is not even a hint of a resonance. For $\zeta = 1$, the undriven system does not oscillate -- the damping is just too great. When the system is driven, it simply follows in a kind of passive, sluggish way.
This response is even more sluggish than the critically damped case. Damping dominates, and the amplitude response falls off even more rapidly with increasing $\gamma$.

For a more systematic view of the dependence on damping for the underdamped case, we now make a movie. Each frame of the movie will show a complete amplitude response curve for a given value of $b$. As the movie progresses, we will see results for ever larger $b$. We choose the values of $b$ so that the damping parameter $\zeta$ varies from 0.01 to 1 in increments of 0.01. To keep the graphs from jumping around because of scale changes, we redefine our graphing routine to use a fixed range of $[0,5]$ for $A_P$. For the smallest values of $\zeta$, the peaks will be truncated by this plot range. As you run the movie, note how the peak rapidly gets lower and broader as the damping increases. Only for very light damping (say $\zeta < 0.1$) do we get sharp resonance peaks and "tuned" systems. The printed version of this notebook shows only 10 graphs.

```
In[32]:= ampresmod[b_] := Plot[amp[b, y], {y, 0, 30},
AxesLabel -> {"y", "A_p"}, PlotRange -> {0, 5}, PlotLabel -> SequenceForm["Amplitude Response for \zeta = ", PaddedForm[N[b/100], {3, 2}]], ImageSize -> 300]

In[33]:= Do[b = 10*i; ampresmod[b], {i, 1, 10}]
```

![Graphs showing amplitude response for different damping levels](ampres.nb)
Amplitude Response for $\zeta = 0.30$

Amplitude Response for $\zeta = 0.40$

Amplitude Response for $\zeta = 0.50$
Amplitude Response for $\zeta = 0.60$

Amplitude Response for $\zeta = 0.70$

Amplitude Response for $\zeta = 0.80$
Amplitude Response for $\zeta = 0.90$

Amplitude Response for $\zeta = 1.00$