ON THE HYDRODYNAMICS OF SUPERFLUID HELIUM

by

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(1958)

SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF

PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF

TECHNOLOGY

September, 1963

Signature of Author  ALFRED CLARK

Department of Mathematics, August 16, 1963

Certified by ___________________________ Thesis Supervisor

Accepted by ___________________________ Chairman, Departmental Committee

on Graduate Students
ABSTRACT

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Submitted to the Department of Mathematics on August 16, 1963 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

The present work is concerned with continuum theories of the hydrodynamics of superfluid helium. A comparative study has been made of some of the existing macroscopic theories of the flow of superfluid helium. The work includes (i) a detailed critical study of the derivations of the hydrodynamic equations in the various theories, (ii) alternative derivations of the equations in some cases and (iii) a discussion of some possible generalizations of some of the existing theories. The theories under consideration include the equations originally proposed by Landau for the hydrodynamics of the two-fluid model, the hydrodynamic equations proposed by Lin for his one-fluid model, the theory of dissipative processes for the two-fluid model proposed by Lin, and the theory of Bekarevich and Khalatnikov.

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ACKNOWLEDGEMENTS

I am greatly indebted to Prof. C. C. Lin for continued guidance and encouragement during the execution of this work. I wish also to acknowledge the valuable assistance of Miss Mary Beth Andre and my wife in the preparation of the final manuscript. I am grateful for the financial support given me during the course of this work by the Massachusetts Institute of Technology and by the National Science Foundation.
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I INTRODUCTION

In the problem of superfluid helium, one may distinguish two main aspects - the development of a molecular theory and the development of a hydrodynamic theory. Ultimately, one would hope to develop the macroscopic hydrodynamic theory from the molecular theory. However, as Lin [25] has emphasized recently, the development of the hydrodynamic theory need not await the resolution of the extremely difficult problems of the molecular theory - one can take advantage of the possibility of developing the macroscopic theory in a more or less phenomenological manner. Of course, the microscopic theory often serves as a guide in the development of macroscopic theories, but the connection at present between the two types of theories is still largely a qualitative one.

In the present work, we consider the problem of developing a macroscopic theory of the hydrodynamics of helium II. In spite of great effort, this problem remains an open question, and a variety of hydrodynamic equations have been proposed. Since this is still a controversial subject, we consider here - on an equal basis - some of the hydrodynamic equations which have been proposed. More specifically, the present work contains (i) a review and criticism of some of the current hydrodynamic theories, with the intent of making more explicit the principles underlying the derivations, (ii) a discussion of some possible generalizations of these theories.
and (iii) a presentation of alternative derivations of the hydrodynamic equations for some of the theories, with the intent of furnishing some new viewpoints. In the remainder of this section, we give a brief review of the hydrodynamic theories to be discussed in more detail later, and a brief discussion of the general features of the problem of determining a set of hydrodynamic equations to describe the flow of helium II.

In obtaining a set of hydrodynamic equations for helium II, one is faced with two fundamental interrelated problems: (i) the selection of a set of (local) macroscopic quantities (e.g., density, velocity, etc.) which will adequately describe the flow of helium II and (ii) the derivation of a set of equations describing the change with time of these quantities. For an ordinary fluid, the problem (i) above presents no difficulties — the macroscopic configuration may be adequately described by mass density, the macroscopic velocity and an additional thermodynamic variable such as the specific entropy. For liquid helium II, however, there is abundant evidence that additional macroscopic quantities are needed to describe the the various flow phenomena; thus the selection of a set of macroscopic quantities to describe the flow of helium II is a matter of some difficulty and importance. Most of the current continuum theories of the flow of helium II take as their starting point some form of the two-fluid model, originated and developed by Landau, London, and Tisza. The qualitative features of this model have been deduced by Landau [20] from
his theory of phonons and rotons, and also by London [28] from
the theory of Bose-Einstein condensation. These alternative
points of view have in common that a definite molecular pic-
ture is used as a guide in the determination of an appropriate
set of macroscopic variables. The recent work of Lin [26,25,
24] has shown, however, that the two-fluid model is by no
means the only starting point for the development of a con-
tinuum theory of helium II, and, also, that even the selection
of an appropriate set of macroscopic variables may be accom-
plished in a phenomenological way without recourse to a defi-
nite molecular picture.

Even after the choice of macroscopic quantities has been
made, there remains the problem of determining a correct ther-
modynamic description of the helium II. Again, there is abun-
dant evidence that the helium II system requires more thermo-
dynamic variables for its description than does an ordinary
fluid. Here, also, a definite molecular picture can be used
as a guide in setting down the fundamental thermodynamic equa-
tion.

Finally, there is the problem of determining the hydro-
dynamic equations. Although the approach to this problem de-
pends to some extent on the choice of macroscopic variables,
there are certain general principles which have been used,
with some success, to obtain a set of hydrodynamic equations.
Among these, we may mention (i) variational principles (for
perfect fluid theories) and (ii) imposition of macroscopic
conservation laws. Of course, analogy with ordinary hydrody-
namics is an important underlying guiding principle in all
approaches to the continuum theory of helium II.

Although there are many problems in the development of a hydrodynamic theory of helium II, there are two issues in particular which are central; the first of these is the question of the rotation of the superfluid component. This is not a question which can be resolved within a given hydrodynamic theory; rather, one must take a definite point of view on this question in order to develop a hydrodynamic theory. In fact, the current theories may be roughly divided into three types with respect to the question of superfluid rotation: (i) theories in which \( \text{curl } \mathbf{v}_s \) is always (Landau [20], Lifshitz and Khalatnikov [22]), (ii) theories in which \( \text{curl } \mathbf{v}_s \) plays a special role (e.g., the quantized vortex line theory of Feynman [7] and Hall and Vinen [12, 13, 14, 38], or the continuum theory of Bekarevich and Khalatnikov [3] in which \( |\text{curl } \mathbf{v}_s| \) is a thermodynamic variable) and (iii) theories in which no special assumptions are made about the nature of \( \text{curl } \mathbf{v}_s \) (Lin [26, 25, 24]).

A theoretical explanation of the problem of superfluid rotation must eventually come from a microscopic theory; one can, however, compare the results of experiments with the predictions of the various hydrodynamic theories and thus gain indirect evidence relevant to the problem. The second central issue is the question of mutual friction (volume momentum exchange) between the two components. Again, one must take a definite point of view on this question in order to develop a hydrodynamic theory.

It is convenient to consider the perfect fluid theories
first, before considering the more general theories including dissipative processes. The principal reason for this is that most of the theories of dissipative processes have as their starting point some form of the perfect fluid equations, so that the development of a perfect fluid theory is a necessary preliminary.

For reversible flows of helium II, the equations proposed by Landau [20, 22] have been well-verified experimentally. As Lifshitz and Khalatnikov [22] have shown, it is possible to deduce Landau's equations by (i) assuming the two-fluid model, (ii) imposing the macroscopic conservation laws and (iii) assuming that the superfluid component must move irrotationally (i.e., $\text{curl} \mathbf{v}_s = 0$). An analysis of their work in detail is given in II-A-1; in particular, it is shown there that the assumption $\text{curl} \mathbf{v}_s = 0$ is a crucial one, in that it is no longer possible to deduce a unique set of equations once it is dropped.

If one takes the point of view that $\text{curl} \mathbf{v}_s$ may vanish for a certain class of solutions, but that it does not necessarily vanish, then the imposition of the conservation laws is not sufficient to determine uniquely the perfect fluid equations for the two-fluid model. However, one may make use of an idea advanced by Landau [20] - namely, that there be no volume momentum exchange between the two components (other than that due to normal fluid-superfluid transitions). In order to make use of this idea, however, one must interpret the two-fluid model in a rather literal fashion and assume that it
is possible to give a separate thermodynamic and hydrodynamic description for each component; the only coupling between the components is through the equilibrium condition for the normal fluid--superfluid transitions. A derivation of the perfect fluid equations along these lines is given in section II-A-2. It is shown there that a perfectly definite set of hydrodynamic equations may be obtained in this manner, and that the equations are identical in form with the Landau equations without, however, the restrictive condition \( \text{curl } \nu = 0 \).

Whenever we are considering a perfect fluid theory, we may expect that some sort of variational principle will be valid. As Lin has shown (see Serrin [33], p. 148, for a discussion of this), there are difficulties with a variational principle even for ordinary hydrodynamics which however may be overcome by taking into account the Lagrangian nature of the system. In the case of the two-fluid model (which does not admit a Lagrangian description), it is not so clear that the difficulties can be resolved. Zilsel [40] has obtained a set of hydrodynamic equations for the two-fluid model by means of a variational principle. His variational principle leads to the Landau equations (with the restrictive condition \( \text{curl } \nu = 0 \)); however, his equations also entail a restriction on the quantity \( \text{curl } \nu \). A detailed discussion of Zilsel's work is given in II-A-3.

A variational principle has also been used by Lin [25, 24] to obtain the perfect fluid equations for his one fluid model. The equations obtained by Lin from his variational principle
are similar to the Landau equations. In Lin’s theory, however, the condition $\text{curl} \mathbf{v}_s = 0$ is characteristic of a particular class of solutions, and for this class his equations are identical with Landau’s. However, Lin’s equations also admit more general solutions for which $\text{curl} \mathbf{v}_s \neq 0$. The variational principle and the equations obtained by Lin are discussed in sections II-B-1 and II-B-2. Since there are at present still some difficulties associated with this variational principle, the possibility of obtaining the perfect fluid equations for the one-fluid model from the conservation laws has been examined. It is shown in section II-B-3 that this method does not lead to a unique set of equations (this, of course, is to be compared with a similar result for the two-fluid model when the restriction $\text{curl} \mathbf{v}_s = 0$ is not imposed).

In Chapters III and IV of the present work, a discussion is given of the hydrodynamic equations including dissipative processes, as proposed by various authors. As in the case of the perfect fluid theories, the form of the final equations depends greatly on the role assigned to the quantity $\text{curl} \mathbf{v}_s$. Even so, the method for deriving the hydrodynamic equations is essentially the same—the basic principles are the macroscopic conservation laws and the principle of increase of entropy. Of course, these principles must be supplemented by other special considerations in each case.

Lifshitz and Khalatnikov [22] have presented an extension of Landau’s theory to include dissipative processes. In the
development of their theory, the restriction \( \text{curl} \, \mathbf{v}_s = 0 \) is imposed. Their equations, along with a possible generalization, are discussed in section III-B.

Since it is known that, under some conditions, the superfluid component seems to rotate in some manner, theories in which \( \text{curl} \, \mathbf{v}_s = 0 \) always are not expected to be of universal validity. Indeed, the experimental discovery that the superfluid can rotate, in conjunction with Landau's idea that \( \text{curl} \, \mathbf{v}_s \) must vanish, has been one of the factors in the development of the quantized vortex line theory. Lin [25, 24, 23] however, has advanced the idea that under certain conditions the superfluid component can rotate in bulk like an ordinary fluid, since at present there seems to be no compelling evidence to the contrary. In the theory of dissipative processes developed by Lin, there is a momentum transfer associated with the superfluid rate of strain as well as the normal fluid rate of strain, and the stress--rate of strain relations are characterized by four shear viscosity coefficients; the nonlinear boundary condition proposed by Lin allows a slip of the superfluid component at a solid wall, while the normal component is assumed to adhere to the wall. A derivation and discussion of Lin's equations is given in section III-C.

The current prevailing theories of the rotation of superfluid helium are based on the Onsager-Feynman theory of quantized vortex lines. Hall and Vinen [12, 13, 14, 38] have developed a set of hydrodynamic equations on the basis of the two-fluid model and the Onsager-Feynman theory. Bekarevich and
Khalatnikov [3] have presented a theory based on continuum principles which, in contrast to the theory of Hall and Vinen, does not depend on the specific features of the quantized vortex line theory. Bekarevich and Khalatnikov based their derivation on the two-fluid model and the single additional assumption that the thermodynamic internal energy of the helium II depends on the magnitude of the superfluid vorticity, as well as the usual thermodynamic variables. They were able to obtain the same final hydrodynamic equations as those obtained by Hall and Vinen. A detailed discussion and criticism of their derivation is given in section IV-B-1. An alternative development of their equations is offered in section IV-B-2. On one hand the theory of Bekarevich and Khalatnikov has the advantage of not resting on a specific molecular picture; on the other hand, its possible relevance to the quantized vortex line theory is not clear. This point is discussed in detail in section IV-B-3. Finally, in section IV-C a possible alternative approach to constructing a hydrodynamic theory including quantized vortex lines is discussed; the equations given by Hall [13] to describe "vortex waves" are obtained there.

In Chapter V, a summary of the various results obtained is given in the form of a unified mathematical scheme which includes the various theories as special cases. Some proposals for further work are also given there.
II PERFECT FLUID THEORIES

A. Two-Fluid Model

1. Equations from conservation laws

Although Tisza presented a set of approximate (linear) hydrodynamic equations for the two-fluid model of helium II in 1938 \cite{35}, the first full set of hydrodynamic equations was presented by Landau in 1941 \cite{20}. Landau's method is briefly described in \cite{20}, and given in more detail by Lifshitz and Khalatnikov \cite{22}. We give in the following a detailed presentation of the derivation of \cite{22}, along with some elaboration of the argument at various points.

According to the two-fluid model, we may (formally) regard helium II as a "mixture" of two fluids - the normal fluid and the superfluid. Each fluid has its own macroscopic velocity and its own mass density. Thus the total mass density \( \rho \) is given by \( \rho = \rho_n + \rho_s \), where \( \rho_n \) and \( \rho_s \) are the normal and superfluid densities, and the momentum per unit volume \( \mathbf{j} \) is given by \( \mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s \), where \( \mathbf{v}_n \), \( \mathbf{v}_s \) are the normal and superfluid velocities, and the entropy per unit volume is given by \( s \), where \( s \) is the entropy per unit mass. Landau \cite{20} has given a microscopic theory of helium II which leads to the two-fluid model in a natural way. In this theory one thinks of the normal fluid as a "gas" of thermal excitations (phonons and rotons), while the superfluid component is the
"inert background" in which the phonons and rotons move. It is clear from this picture that the macroscopic theory must allow for the possibility of conversion of normal fluid into superfluid (and vice versa).

Then the laws of conservation of mass, momentum and entropy will have the forms

\[ \frac{\partial \rho}{\partial t} + \text{div}(\mathcal{J}_p) = 0, \]  
(1)

\[ \frac{\partial j_j}{\partial t} + \frac{\partial}{\partial x_j}(\mathbf{m}_{ij}) = 0, \]  
(2)

\[ \frac{\partial}{\partial t}(\rho s) + \text{div}(\mathcal{J}_s) = 0, \]  
(3)

where \( \mathcal{J}_p \) is the mass flux vector, \( \mathcal{J}_s \) the entropy flux vector and \( \mathbf{m}_{ij} \) the momentum flux tensor. We must also require the conservation of energy, which is expressed by

\[ \frac{\partial E}{\partial t} + \text{div} \mathcal{Q} = 0, \]  
(4)

where \( E \) is the total energy per unit volume and \( \mathcal{Q} \) is the energy flux vector. To complete the set of equations, one additional (vector) equation is needed, and we may take this to be the equation for the superfluid velocity \( \mathbf{v}_s \). This may be written as

\[ \frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s = \mathbf{f}. \]  
(5)
Equations (1) - (5) would be a complete set of hydrodynamic equations if the dependence of the quantities $f_p$, $f_s$, $Q$, $\Pi_{ij}$ and $f$ on the quantities $\rho_n$, $\rho_s$, $S$, $\nu_n$ and $\nu_s$ were known, and if the dependence of the total energy $E$ on $\rho_n$, $\rho_s$, $\nu_n$, $\nu_s$ and $S$ were known. These equations represent 9 scalar equations for the 9 scalar quantities $\rho_n$, $\rho_s$, $S$, $\nu_n$, $\nu_s$, $\nu_{nl}$, $\nu_{sl}$ and it would seem that any choice of $f_p$, $f_s$, $Q$, $\Pi_{ik}$ and $f$ (consistent with the Galilean transformation laws) would lead to a set of hydrodynamic equations. This is not the case, however, because of the following essential feature of the two-fluid model: the 9 quantities $\rho_n$, $\rho_s$, $S$, $\nu_n$, $\nu_s$, $\nu_{nl}$, $\nu_{sl}$ are not independent, but satisfy a sort of equilibrium relation. This conclusion is a consequence of the possibility of normal fluid - superfluid transitions mentioned above. Thus we must require that the 9 scalar equations (1) - (5) for the 8 independent quantities be consistent, and this leads to a restriction on the "fluxes" $f_p$, $\Pi_{ij}$, $f_s$, $Q$ and $f$. Preliminary to exploiting this fact, however, we must examine the dependence of the energy $E$ on the quantities $\rho_n$, $\rho_s$, $S$, $\nu_n$, $\nu_s$ and also the nature of the equilibrium relation.

The energy $E$ should be expressible as a function of 8 independent quantities. It is convenient to choose the quantities $\rho$, $\nu$, $S$ and $\nu_s$ as the independent ones. It is to be expected that $\rho_n$ (for example) will be a definite function of these 8 independent quantities. (Although it would seem that any choice of 8 quantities could be used, the discussion
of part II-A-2 will make clear the significance of this particular choice). By means of the Galilean transformation for energy, we may express the energy \( E \) in terms of the energy \( E_0 \) as measured in the superfluid rest frame; thus

\[
E(p, s, \vec{j}, \nu_s) = E_0(p, s, \vec{j}_0) + \vec{j}_0 \cdot \nu_s + \frac{p \nu_s^2}{2}
\]  

(6)

where \( \vec{j}_0 \) is the momentum per unit volume as measured in the superfluid rest frame, and \( E_0(p, s, \vec{j}_0) = E(p, s, \vec{j}_0, 0) \). The momentum \( \vec{j}_0 \) is related to \( \vec{j} \) by the Galilean transformation formula

\[
\vec{j} = p \nu_s + \vec{j}_0.
\]  

(7)

\( E_0 \) may depend on \( \vec{j}_0 \) only through the scalar quantity \( \frac{1}{2} \nu_s^2 \), so that

\[
E_0 = f(p, s, \frac{1}{2} \nu_s^2).
\]  

(8)

The function \( f \) cannot be determined by continuum principles, but should (in principle) follow from a detailed microscopic theory. According to Landau's view of a gas of excitations (the normal fluid) moving in an inert background (the superfluid), the observer moving with the inert background (i.e., in the superfluid rest frame) should see something quite similar to the motion of an ordinary gas with a drift velocity \( \vec{w} = \nu_n - \nu_s \). This qualitative picture is a basis for the assumption that the intensities corresponding to the quantities \( p \), \( s \), \( \vec{j}_0 \) are the same as those for an ordinary fluid. For an
ordinary fluid of energy $E$ per unit volume, density $\rho$, entropy $s$ per unit volume and momentum $j$ per unit volume, the differential of $E$ is given by $dE = \nabla \cdot j + Td(ps) + \Phi d\rho$, where $\nabla$ is the velocity of motion, $T$ the temperature and $\Phi$, the thermodynamic potential is related to the pressure $p$ by $\rho \Phi = E - \nabla \cdot j - Ts + p$. Thus we assume that the differential of $E_o$ has the form

$$dE_o(\rho, s, j_o) = \Phi d\rho + Td(ps) + \nabla \cdot d\rho$$

where

$$\rho \Phi = E_o - \nabla \cdot j_o - Ts + p.$$  

From (8), we have that $$(\frac{\partial E_o}{\partial j_o})_{\rho, s} = \frac{1}{\rho} (\frac{\partial \nabla}{\partial j_o})_{\rho, s};$$ since $j_o = \rho_n \nabla$, we see that the normal fluid density is given by

$$\rho_n = \frac{1}{(\frac{\partial \nabla}{\partial [\frac{1}{2} j_o^2]})_{\rho, s}},$$

and we may solve (11) to obtain $\rho_n$ as a function of $\rho$, $s$ and $\frac{1}{2} \nabla^2$. Thus the specification of the function $E_o(\rho, s, j_o)$ determines both the dependence of the total energy on the independent macroscopic quantities, and the dependence of the normal fluid density on $\rho$, $s$, $\frac{1}{2} \nabla^2$.

For purposes of comparison with other formulations of the two-fluid thermodynamics, it is convenient to introduce the internal energy per unit mass $e$, defined by

$$E = \frac{1}{2} \rho_n v_n^2 + \frac{1}{2} \rho_s v_s^2 + \rho e.$$
Then one may show from (6), (9) and (10) that

$$\rho e = E_0 - \frac{1}{2} \rho_n w^2, \quad (13)$$

and

$$\delta e = \frac{\rho}{\rho_n} \delta \rho + T \delta s + \frac{1}{2} w^2 \delta x, \quad (14)$$

where

$$x = \frac{\rho_n}{\rho} \quad (15)$$

is the normal fluid concentration. Thus the natural independent variables for $e$ are $\rho, s$ and $x$ (instead of $\rho, s, \frac{1}{2} j^2$), and the relation analogous to (11) is

$$\left( \frac{\partial e}{\partial x} \right)_{e,s} = \frac{1}{2} w^2. \quad (16)$$

Some comment about equation (10) (which may be regarded as a definition of the pressure $\rho$) is perhaps in order. First, one may show from (6) - (10) that the derivative of the total energy in a volume $\gamma$ with respect to $\gamma$ (at fixed total momentum, entropy and mass, and fixed superfluid velocity) is $-\rho$; second, we shall see that the quantity $\rho$ does indeed play the role of a normal stress in the hydrodynamic equations.

Although the arguments leading to equation (9) (and, in particular, to the relation $\left( \frac{\partial E_0}{\partial s} \right)_{e,s} = \gamma$) are perhaps not compelling, it will be shown later (sections II-A-2 and II-A-3) that different approaches to the determination of the thermodynamic description of helium II lead to the same results.

The remaining problem in the derivation of the hydrodynamic equations is the determination of the quantities $\mathcal{E}_p$,

$\mathcal{E}_{ij}$, $\mathcal{F}_s$, $\mathcal{Q}$ and $\mathcal{f}$. 
By analogy with ordinary hydrodynamics, we expect that the mass flux $\mathcal{F}_m$ may be identified with the momentum density $\mathbf{j}$; thus we assume

$$\mathcal{F}_m = \mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s.$$  \hfill (17)

This, of course, is consistent with a very literal interpretation of the two-fluid model. However, there does remain the possibility that $\mathcal{F}_m \neq \mathbf{j}$; in fact, Lee and Yang [21] have derived, for a dilute system of hard sphere bosons, a set of transport equations which exhibit this possibility. Since much of the "physical feeling" for the macroscopic quantities of the two-fluid model is rooted in the concepts of ordinary hydrodynamics, the introduction of a mass flux $\mathcal{F}_m \neq \mathbf{j}$ would require a critical re-evaluation of the physical significance of the quantities $\rho_n$, $\rho_s$, $\mathbf{v}_n$ and $\mathbf{v}_s$. In the following we will always assume (17) to hold.

It is usually assumed in the two-fluid model that the total entropy resides in the normal fluid; in view of the microscopic picture of the normal fluid as a gas of thermal excitations, this is a natural assumption to make. In the perfect fluid theory for the two-fluid model, the only mechanism for entropy transport is convection; thus we take the entropy flux $\mathcal{F}_s$ to be

$$\mathcal{F}_s = \rho_s \mathbf{v}_n.$$  \hfill (18)

We now consider the determination of the fluxes $\mathbf{m}_n$ and
We note first that the conservation of angular momentum (assuming that the angular momentum density is $\pi \times j$) requires that $\pi j$ be symmetric. Since the conservation equations must be invariant in form under Galilean transformations, and since the transformation properties of $E$ and $j$ under such a transformation are known (cf. (6) and (7)), the transformation properties of the fluxes $\pi j$ and $Q$ may be easily deduced. We may use these transformation formulae to express $\pi j$ and $Q$ in terms of $\pi j^o$, the momentum flux as seen in the superfluid rest frame, and $Q^o$, the energy flux as seen in the superfluid rest frame. Thus

$$\pi ij = e v si v sj + v si j o j + v sj j o i + \pi ij^o \tag{19}$$

$$Q = \left(\frac{e v s^2}{2} + j o \cdot v s + E o\right) v s + \frac{v s^2}{2} j o + (\pi o \cdot v s) + Q^o \tag{20}$$

where $\pi o \cdot v s$ denotes the vector with components $\pi ij^o v sj$. The advantage of this is that $\pi ij^o$ and $Q^o$, being Galilean invariants, can depend on $v o$ and $v s$ only through the difference $w = v o - v s$. Thus the problem is to determine $\pi ij^o$, $Q^o$ and the function $f$ in equation (5).

The range of possibilities for the function $f$ depends greatly on whether or not the superfluid can rotate. According to Landau's original view [20], the superfluid flow must be irrotational, which means that $f$ must have the form

$$f = \nabla \phi \tag{21}$$

According to the later work of Feynman [7] and Hall and Vinen
[12,13,14,38], the superfluid remains point-wise irrotational but imitates bulk rotation by means of quantized vortex lines. In the theories of Hall and Vinen, the rotation is held to be closely connected with the dissipative phenomenon of mutual friction. According to Lin's theory [26, 25, 24], irrotational motion of the superfluid is a possible solution of the perfect fluid equations, but the equations also allow more general solutions for which \( \text{curl} \mathbf{v} \neq 0 \). A detailed discussion of the question of superfluid rotation has been given by Lin [25]. Since this is still an unsettled question, we consider both possibilities here.

We examine first the case when the superfluid flow must be irrotational; then equation (21) must hold, and the problem is reduced to the determination of the quantities \( \phi \), \( \pi_{ij}^0 \) and \( \mathcal{Q}^0 \). As mentioned earlier, the 9 (scalar) equations represented by (1) - (5) are not independent, and we must require that they be consistent. To impose this requirement, we first calculate \( \frac{\delta E}{\delta t} \) in terms of \( \frac{\delta \rho}{\delta t}, \frac{\delta \psi}{\delta t}, \frac{\delta v_i}{\delta t} \) and \( \frac{\delta j}{\delta t} \) by means of (6) and (7); then we may express these last time derivatives in terms of the fluxes by means of (1) - (3) and (5). Finally we substitute the expression for \( \frac{\delta E}{\delta t} \) into (4), and the resulting equation must be an identity, since it contains no time derivatives. It is convenient to express the final result in terms of \( \pi_{ij}', \mathcal{Q}' \) and \( \phi' \), where

\[
\pi_{ij}' = \rho \delta_{ij} + j_i w_j + \pi_{ij}^0',
\]

\[
\mathcal{Q}' = \phi_j w_j + \nabla \psi w + j \omega w^2 + Q',
\]
and \( \Phi = -\Phi + \Phi' \).

The result is

\[ \text{div} \left\{ Q' - \Pi^o' \cdot w \right\} + \Pi^o_k \frac{\partial \gamma_i}{\partial x_k} + (j - \rho \gamma_n) \cdot \nabla \Phi' = 0. \tag{25} \]

One obvious "solution" of (25) is \( Q' = 0 \), \( \Pi^o = 0 \) and \( \Phi' = 0 \); however, one can also find non-zero solutions, so some further information is needed in order to obtain a unique final answer. In principle, the quantities \( \Pi^o_j \), \( Q' \) and \( \Phi' \) may depend on \( \rho \), \( \rho s \) and \( w \), as well as spatial derivatives of all of the independent macroscopic variables. In a perfect fluid theory, however, one might expect that the fluxes will not depend on the gradients of the macroscopic variables, although there seems to be no compelling reason to believe this. In any case, the "simplest" set of hydrodynamic equations will correspond to this case, so we simply assume that \( \Pi^o_j \), \( Q' \) and \( \Phi' \) (and consequently \( \Pi^o_j' \), \( Q' \) and \( \Phi' \)) depend only on the relative velocity \( w \) (by Galilean invariance, these quantities cannot depend on \( \gamma_o \) and \( \gamma_s \) separately) and on the thermodynamic quantities, and not on the spatial derivatives of these quantities. Then, from the first and third terms in (25), only gradients of \( w_i \) - and not of \( \gamma_i \) alone - can arise; since \( \Pi^o_k \) is to be independent of these gradients, it must be that \( \Pi^o = 0 \). Then (25) takes the form

\[ \text{div} Q' + (j - \rho \gamma_n) \cdot \nabla \Phi' = 0, \]
or

\[ \text{div} \mathbf{Q}' - \epsilon_s \mathbf{w} \cdot \nabla \phi' \equiv 0. \]  

(26)

Since \( \mathbf{w} \) is the only vector on which \( \mathbf{Q}' \) may depend, \( \mathbf{Q}' = \lambda \mathbf{w} \), where \( \lambda \) is a Galilean invariant scalar function. A straightforward analysis then shows that \( \lambda \equiv 0 \), and \( \phi' \equiv \text{constant} \) (which we may take as zero). Thus under the assumptions made, we obtain a unique set of hydrodynamic equations. The final expressions for the fluxes may be written as

\[ \Pi_{ij} = \rho_n v_{ni} v_{nj} + \rho_s v_{si} v_{sj} + p \delta_{ij}, \]  

(27)

\[ \mathbf{Q} = (\mathbf{\Phi} + \frac{1}{2} \mathbf{v}^2) j + T \rho_s v_n + \rho_n v_n (\mathbf{w} \cdot \mathbf{v}_n), \]  

(28)

\[ f = -\nabla \Phi. \]  

(29)

For purposes of comparison with other derivations, we write out the final equations here; for convenience, the internal energy \( e \) (cf. (13)) is used in place of \( E_0 \). The equations are

\[ \frac{\partial e}{\partial t} + \text{div} (\rho_n v_n + \rho_s v_s) = 0, \]  

(30)

\[ \frac{\partial}{\partial t} (\rho_n v_n + \rho_s v_s) + \text{div} (\rho_n v_n v_n + \rho_s v_s v_s) = -\nabla p, \]  

(31)
entropy: \[ \frac{\partial}{\partial t}(\rho s) + \text{div} (\rho s v_s) = 0, \quad (32) \]

and the superfluid equation: \[ \frac{\partial v_s}{\partial t} = -\nabla \left( \frac{1}{2} v_s^2 + \Phi \right), \]
or
\[ \frac{\partial v_s}{\partial t} + v_s \nabla v_s = -\nabla T - \nabla p + x \nabla \frac{1}{2} w_s^2 \quad (33) \]
and the restrictive equation \( \text{curl} \, v_s = 0 \).

These equations imply the energy equation, which may be written as
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho_n v_n^2 + \frac{1}{2} \rho_s v_s^2 + \rho e \right) + \text{div} \left( \frac{1}{2} \rho_n v_n^2 v_n + \frac{1}{2} \rho_s v_s^2 v_s + \rho e v \right) + \frac{1}{2} \rho_s v_s^2 v_s + (\rho e + p) v + T \rho_s (v_0 - v) + \rho x w_s^2 (v_0 - v) \right) = 0, \quad (35) \]

where we have introduced the mass velocity \( v = x y_0 + (1-x) y_s = \frac{j}{t} \).

The energy equation may be put into another form which clearly shows the physical significance of the various terms (Zilsel [30]; London [29]):
\[ \frac{\partial}{\partial t} \left( \frac{1}{2} \rho_n v_n^2 + \frac{1}{2} \rho_s v_s^2 + \rho e \right) + \text{div} \left[ \frac{1}{2} \rho_n v_n^2 v_n + \frac{1}{2} \rho_s v_s^2 v_s + \rho e v \right] + \rho_s \left( \frac{\partial \rho}{\partial s} \right)_{\rho_s} (v_0 - v) + \rho x \left( \frac{\partial e}{\partial s} \right)_{\rho_s} (v_0 - v) \right] = - \text{div} (p v). \quad (36) \]

The first three terms in the energy flux \( \frac{1}{2} \rho_n v_n^2 v_n + \frac{1}{2} \rho_s v_s^2 v_s + \rho e v \)
represent the convection of energy by the motion; the last two terms give a correction to the term \( p \mathbf{e} \mathbf{v} \), this correction being needed because the entropy and normal fluid concentration follow the motion of the normal fluid rather than the bulk motion. Finally, we may obtain an equation for the normal fluid by combining the mass, momentum and superfluid equations. The result is

\[
\frac{\partial \mathbf{v}_n}{\partial t} + \mathbf{v}_n \cdot \nabla \mathbf{v}_n = -\frac{1}{\rho} \nabla p - \frac{1-x}{x} s\nabla T - (1-x) \nabla \mathbf{v}^2 \cdot \mathbf{w} \cdot \frac{\Gamma}{\rho}, \tag{37}
\]

where

\[
\Gamma = \frac{\partial \rho_n}{\partial t} + \text{div} (\rho_n \mathbf{v}_n) \tag{38}
\]

and is the volume rate of conversion. (\( \Gamma \) of course cannot be independently specified, but is determined by the spatial distribution of the independent quantities; in fact one can obtain an explicit (though not very enlightening) expression for \( \Gamma \).)

We consider now the boundary conditions to be satisfied at a solid-helium II interface. For convenience, we assume that the solid is at rest (which we may accomplish by an appropriate Galilean transformation). First of all, we must require that there be no mass flux across the interface; thus

\[
\mathbf{j} \cdot \mathbf{n} \bigg|_{\text{wall}} = \rho_n (\mathbf{v}_n \cdot \mathbf{n}) + \rho_s (\mathbf{v}_s \cdot \mathbf{n}) \bigg|_{\text{wall}} = 0 \tag{39}
\]

or

\[
x (\mathbf{v}_n \cdot \mathbf{n}) = -(1-x)(\mathbf{v}_s \cdot \mathbf{n}) \quad \text{at the wall.}
\]
If there is no energy transport across the wall, then we must also have \( \mathbf{Q} \cdot \mathbf{n} = 0 \) at the wall. (Equation (39)) of course holds whether or not there is an energy transport.) This, in conjunction with (39), leads to \( \mathbf{v}_n \cdot \mathbf{n} = \mathbf{v}_s \cdot \mathbf{n} = 0 \). Finally, it is expected that the temperature will be continuous in this case. Thus

\[
\mathbf{v}_n \cdot \mathbf{n} \bigg|_{\text{wall}} = 0, \quad \text{(case of no energy transport)}
\]

and

\[
\mathbf{v}_s \cdot \mathbf{n} \bigg|_{\text{wall}} = 0, \quad (40)
\]

The discussion of the boundary conditions for the case of non-zero energy transport is complicated by a peculiar theoretical difficulty which is best illustrated by an example. Suppose there is a uniform heat flux of magnitude \( \mathbf{H} \) from the solid to the helium II (see Fig. 1). Then by conservation of energy, we have

\[
\mathbf{Q} \cdot \mathbf{n} = \mathbf{H} \cdot \mathbf{n}
\]

which, by using (39), we may write as

\[
(\mathbf{v}_n \cdot \mathbf{n}) \left\{ \frac{C}{1-x} (\mathbf{v}_0 \cdot \mathbf{n})^2 + C \sigma \mathbf{T}_{\text{fluid}} \right\} = \mathbf{H} \cdot \mathbf{n}, \quad (41)
\]

where for simplicity, we have assumed that the tangential component of \( \mathbf{v}_0 \) vanishes. Since \( \mathbf{H} \cdot \mathbf{n} > 0 \) and since the quantity in brackets is always positive, we have \( \mathbf{v}_s \cdot \mathbf{n} > 0 \). Now consider
Figure 1
the entropy flux in the solid; this is simply \( \frac{H\cdot n}{T_{\text{wall}}} \). The entropy flux in the fluid is given by \( \rho s (v_n \cdot n) \). By the second law of thermodynamics, the entropy flux in the fluid must be at least as large as that in the solid; hence

\[
\rho s (v_n \cdot n) \geq \frac{H\cdot n}{T_{\text{wall}}} = \left( \frac{v_n \cdot n}{T_{\text{wall}}} \right) \left\{ \frac{\varepsilon}{1-x} (v_n \cdot n)^2 + \rho s T_{\text{fluid}} \right\},
\]

or, since \( v_n \cdot n > 0 \), we may write this as

\[
T_{\text{wall}} - T_{\text{fluid}} \geq \frac{x}{(1-x)} \frac{(v_n \cdot n)^2}{s} \left( \frac{\text{net energy flux}}{\text{fluid}} \right) \text{ (directed into solid)} \tag{42}
\]

It would appear that the temperature of the wall must be greater than that of the fluid. The reason for (41) is clear: some of the thermal energy in the solid is converted into what is mechanical energy in the helium; if the temperatures were equal, we would have a decrease of entropy. (It should be pointed out that the actual magnitude of the effect is extremely small; if, for example, \( T_{\text{fluid}} = 1.8^\circ \text{K} \) and \( v_n \cdot n = 10 \text{ cm/sec} \), then \( x \approx \frac{1}{8} \), \( s \approx 5.35 \times 10^6 \text{ ergs g}^{-1} \text{m}^{-2} \text{deg}^{-1} \), and (42) gives \( T_{\text{wall}} - T_{\text{fluid}} \approx 10^6 \text{K} \). If the net energy transport is from the fluid into the solid, then the second law requires only that

\[
T_{\text{wall}} - T_{\text{fluid}} \leq \frac{x}{(1-x)} \frac{(v_n \cdot n)^2}{s} \left( \frac{\text{net energy flux}}{\text{flux directed into solid}} \right) \tag{43}
\]

and, in particular, we may have \( T_{\text{wall}} = T_{\text{fluid}} \). The result (42) means that we cannot specify \( T_{\text{wall}} = T_{\text{fluid}} \) in the case of a heat flux into the fluid, and in fact, it is not clear on the basis
of the perfect fluid theory what the boundary conditions on $T_{\text{fluid}}$ should be. However, it is possible to resolve this question by means of the more general theories which include dissipative processes, and this analysis will be given in detail in part III. For now we only note that in almost all cases of practical interest, the kinetic energy terms in the energy flux are negligible, and we may use a set of effective boundary conditions of the form

$$x(v_n \cdot n) + (1-x)(v_s \cdot n) = 0,$$

$$\rho = T_{\text{fluid}} (v_n \cdot n) = H \cdot n,$$

$$(T_{\text{wall}} - T_{\text{fluid}}) = A(H \cdot n),$$

where the thermal resistance $A$ depends on the fluid temperature $T_{\text{fluid}}$.

In the perfect fluid theory, there are, of course, no conditions imposed on the tangential components of $v_n$ and $v_s$.

This completes the presentation of Landau's derivation of the hydrodynamic equations for the two-fluid model. Although the method can be extended to the case of irreversible processes, a discussion of this will be deferred to part III. It is perhaps well to explicitly list the assumptions and principles which were necessary in order to obtain a definite set of hydrodynamic equations: (i) the two-fluid model (with densities $\rho_n$, $\rho_s$ and velocities $v_n$, $v_s$, together with the (usual) assumption that the superfluid component carries no
entropy), (ii) the imposition of conservation laws and the
Galilean relativity principle, (iii) an assumption about the
form of the thermodynamic energy function \( E \), (iv) the as-
sumption that \( \text{curl} \, \mathbf{v}_s = 0 \) always, and (v) the assumption that the
fluxes do not depend on the gradients of the fundamental
macroscopic variables. If the assumption \( \text{curl} \, \mathbf{v}_s = 0 \) is dropped,
then it is no longer possible to obtain a definite set of hy-
drodynamic equations. We give a brief discussion of the re-
sults in this case below.

If it is no longer required that \( \text{curl} \, \mathbf{v}_s \) must be zero,
then equation (21) no longer is necessarily true. It is still
convenient to introduce the quantities \( \pi_{ij} \) and \( q' \) (equations
(22) and (23)), and it is convenient to introduce

\[
\mathbf{f} = - \nabla \Phi + \mathbf{f}'.
\]  

(45)

The calculations are exactly as before, and the restrictive
equation corresponding to (25) is

\[
\text{div} \left\{ Q' - \pi' v_n \right\} + \pi_{ij}' \frac{\partial v_i}{\partial x_j} + (j - \rho v_n) \cdot (\mathbf{f}' + [\text{curl} \, \mathbf{v}_s] \times \mathbf{w}) = 0.
\]  

(46)

Since \( j + \rho v_n = \rho_s w \), this may be written as

\[
\text{div} \left\{ Q' - \pi' v_n \right\} + \pi_{ij}' \frac{\partial v_i}{\partial x_j} - \rho_s w \cdot \mathbf{f}' = 0.
\]  

(47)

In the case of \( \text{curl} \, \mathbf{v}_s = 0 \), we have \( \mathbf{f} = - \nabla \Phi \), where \( \Phi \) depends
only on thermodynamic quantities (and the relative velocity \( \mathbf{w} \)); \( \mathbf{f} \), however, depends on the spatial derivatives of
these quantities so we must expect that, in the case $\text{curl} \mathbf{v}_s \neq 0$
the "force" $\mathbf{f}'$ in general will depend on the gradients of the macroscopic variables.

The equation (41) allows a wide variety of "solutions". In particular, we see that it is satisfied for $\pi^{'0}_{ij} \equiv 0$, $Q' \equiv 0$
and $\mathbf{f}' = \mathbf{w}_s \times \mathbf{e}$, where $\mathbf{e}$ is an arbitrary (Galilean invariant) vector. (If we choose $\mathbf{e} = \times \text{curl} \mathbf{v}_s$ so that $\mathbf{f}' = \times \mathbf{w}_s \times \text{curl} \mathbf{v}_s$, the
resulting equations are identical with those obtained by Lin [25, 24, 26]; these equations will be discussed in section II-B). The appearance of only the normal fluid rate of strain tensor in (41) is somewhat misleading, since it is merely a consequence of the choice of notation. In fact, if we decompose the (as yet unknown) functions $\pi^{'0}$ and $\mathbf{f}'$ as follows:

$$\pi^{'0}_{ij} = \pi^{'n}_{ij} + \pi^{'s}_{ij},$$

$$p_s \mathbf{f}' = \mathbf{F} - \text{div} \mathbf{\pi}^{'s},$$

then equation (47) may be written in the more symmetric form

$$\text{div} \left\{ Q' - \pi^{'n}_{ij} \mathbf{v}_n - \pi^{'s}_{ij} \mathbf{v}_s \right\} + \pi^{'0}_{ij} \frac{\partial \mathbf{v}_i}{\partial x_j} + \pi^{'s}_{ij} \frac{\partial \mathbf{v}_i}{\partial x_j} - \mathbf{w}_s \cdot \mathbf{F} \equiv 0. \quad (49)$$

By way of example, we give the following specification of $Q'$, $\pi^{'n}$, $\pi^{'s}$ and $\mathbf{F}$, which satisfies (49):

$$Q' = \pi^{'n}_{ij} \mathbf{v}_n + \pi^{'s}_{ij} \mathbf{v}_s,$$

$$\pi^{'n}_{ij} = \beta_n \mathbf{w}_i \mathbf{w}_j, \quad \pi^{'s}_{ij} = \beta_s \mathbf{w}_i \mathbf{w}_j,$$  

(50)
and \[ F = \beta_n \mathbf{w} \cdot \nabla \mathbf{v}_n + \beta_s \mathbf{w} \cdot \nabla \mathbf{v}_s \]

(where \( \beta_n, \beta_s \) are arbitrary scalar functions of the thermodynamic variables and of \( \mathbf{w}^2 \)).

The above examples show clearly that one needs some additional information in the case \( \text{curl} \mathbf{v}_s \neq 0 \) in order to obtain a definite set of hydrodynamic equations. Actually, it is not so surprising that there should be some difficulty in deducing a unique set of hydrodynamic equations from conservation laws, because we cannot really expect to obtain any information about the momentum transfer between the two components from the fact that the total momentum is conserved.

Although it was a part of Landau's original theory [20] that there should be no momentum exchange between the two components (other than that due to \( \mathbf{v}_n \leftrightarrow \mathbf{v}_s \) transitions), did not need to appeal to this principle in his development of the hydrodynamic equations, since the conservation laws plus the requirement \( \text{curl} \mathbf{v}_s = 0 \) uniquely determined these equations. In the next section we will show that if the requirement \( \text{curl} \mathbf{v}_s = 0 \) is dropped, one may still obtain a unique set of hydrodynamic equations by making use of this principle of no momentum exchange. Furthermore, the resulting equations are identical in form with Landau's equations (without the restriction \( \text{curl} \mathbf{v}_s = 0 \)).
2. Equations from formal analogy with ordinary hydrodynamics

In this section we will give an alternative derivation of the perfect fluid equations; the final equations obtained are the Landau equations without the restriction \( \text{curl} \mathbf{v}_s = 0 \). In this way the Landau equations—which have been well-verified experimentally—are freed from the assumption that the superfluid cannot rotate. The derivation is based primarily on analogy with ordinary hydrodynamics, and also the principle that there should be no momentum exchange between the two components; appeal is also made to certain qualitative ideas from the microscopic theory. It will be seen that the point of view taken here affords a particularly simple characterization of Landau's hydrodynamic equations.

The starting point for the derivation is the two-fluid model; we assume that the helium may be regarded as a "mixture" of two fluids; the normal fluid has density \( \rho_n \), velocity \( \mathbf{v}_n \), and entropy \( \xi_n \) per unit volume, while the superfluid has density \( \rho_s \), velocity \( \mathbf{v}_s \) and no entropy (the possibility of superfluid entropy is easily included in the following theory; however, there are no experiments as yet to indicate such a generalization). We assume that each of the fluids has a thermodynamic internal energy function, so that if \( \varepsilon_n \), \( \varepsilon_s \) are the total energies per unit volume of the normal fluid and superfluid respectively, we have

\[
\varepsilon_n = \frac{1}{2} \rho_n \mathbf{v}_n^2 + \rho_n \xi_n
\]
and 

\[ \varepsilon_s = \frac{1}{2} \rho_s v_s^2 + \rho_s \varepsilon_s, \]  

(51)

where \( \varepsilon_n \) is the internal energy of the normal fluid, per unit mass of normal fluid, and similarly for \( \varepsilon_s \). We expect that \( \varepsilon_n \) will be a function of \( \rho_n, s_n \), and that the normal fluid pressure \( p_n \) and the normal fluid temperature \( T_n \) are given by

\[ \frac{d\rho_n}{\rho_n} = \frac{p_n}{\rho_n} \frac{d\varepsilon_n}{\varepsilon_n} + T_n ds_n. \]  

(52)

Since the superfluid component has no entropy, \( \varepsilon_s \) will be a function of \( \rho_s \) only, and the superfluid pressure \( p_s \) is given by

\[ \frac{d\rho_s}{\rho_s} = \frac{p_s}{\rho_s} \frac{d\varepsilon_s}{\varepsilon_s}. \]  

(53)

The existence of the functions \( \varepsilon_n, \varepsilon_s \) and the equations (52), (53) are simply assumed on the basis of analogy with the thermodynamics of ordinary substances.

Suppose now we consider the total energy \( \varepsilon \) in a fixed volume \( v \); \( \varepsilon \) will be given by

\[ \varepsilon = (\varepsilon_n + \varepsilon_s) v = \varepsilon v \]  

(54)

The total momentum \( \mathbf{j} \), mass \( M \) and entropy \( S \) in \( v \) are given by

\[ \mathbf{j} = v (j_n + j_s), \quad (j_n = \rho_n v_n, j_s = \rho_s v_s), \quad M = v (\rho_n + \rho_s), \]

and \( S = v \rho_n s_n \).

(55)
If the two fluids are in equilibrium, we would expect that the distribution of the total momentum and the total mass over the two components would be given by an energy minimum principle, namely

\[
\frac{d \epsilon}{d \mathcal{I}, \mathcal{M}, \mathcal{S}, \mathcal{V}} = 0 .
\]

(56)

This equation leads to the results

\[
\nu_n = \nu_s ,
\]

(57)

and

\[
\Phi_n = \Phi_s ,
\]

(58)

where we have introduced the thermodynamic potentials \( \Phi_n \) and \( \Phi_s \), defined by

\[
\Phi_n = \epsilon_n - T S_n + \frac{p_n}{\epsilon_n} , \quad \text{and} \quad \Phi_s = \epsilon_s + \frac{p_s}{p_s} .
\]

(59)

Equation (57) would seem to say that the two fluids must move together when in equilibrium. In fact, Lee and Yang [21] have emphasized the point of view that the states for which \( \nu_n \neq \nu_s \) are in reality quasi-equilibrium states with a macroscopic lifetime. The problem, then, is to modify the equilibrium condition (56) so as to allow for such states; to do this, we borrow some qualitative ideas from the microscopic theory (22) and [21]). The key point is that the superfluid velocity \( \nu_s \) is not an average over a thermal distribution, and a change in \( \nu_s \) requires a coherent change in the state of a large number of particles; thus we may take \( \nu_s \) to be an invariant of a microscopic collision. The collisions of course also conserve momentum, mass and energy, so that the natural
thermodynamic description is to regard the entropy as a function of the momentum, mass, energy and superfluid velocity \( v_s \), or, equivalently, to regard the energy as a function of the entropy, mass, momentum and superfluid velocity. From this point of view, then, it is natural to take \( J, \rho, S, v_s \), as the eight independent macroscopic quantities describing the motion. We may then expect that the modified equilibrium principle takes the form

\[
\frac{d \epsilon}{J, S, v, v_s} = 0, \tag{60}
\]

and this leads to the single equilibrium condition

\[
\Phi_n - \Phi_s = \frac{1}{2} \omega^2. \tag{61}
\]

If we again introduce the total internal energy per unit total mass, \( \epsilon \), by

\[
\epsilon_n + \epsilon_s = \frac{1}{2} \rho_n v_n^2 + \frac{1}{2} \rho_s v_s^2 + \rho \epsilon \tag{62}
\]

or

\[
\rho \epsilon = \rho_n \epsilon_n + \rho_s \epsilon_s,
\]

and the entropy \( S \) per unit total mass by

\[
\rho_n S_n = \rho S \tag{63}
\]

then, from (52), (53) and (61), we have

\[
de = T dS + \left( \frac{P_n + P_s}{\rho^2} \right) d\rho + \frac{1}{2} \omega^2 dx, \tag{64}
\]
and, if we introduce the total pressure \( p \) by

\[
P = p_n + p_s,
\]

(65)

(64) becomes

\[
de = T ds + \frac{p}{\rho^2} \, d\rho + \frac{1}{2} w^2 \, dx.
\]

(66)

Thus we have arrived at the same thermodynamic description as that given in the last section; however, the above derivation emphasizes the nature of the equilibrium relation and the distinctive nature of the superfluid velocity \( v_s \). For later convenience, we note here that the differentials of the partial pressures \( p_n \), \( p_s \) are given by

\[
\frac{dp_n}{p_n} = \frac{dp}{\rho} + \frac{(1-x)}{x} \, s \, dT + (1-x) \, d\left(\frac{1}{2} w^2\right),
\]

(67)

\[
\frac{dp_s}{p_s} = \frac{dp}{\rho} - s \, dT - x \, d\left(\frac{1}{2} w^2\right).
\]

It should be pointed out that the development of the thermodynamics to this point could also have been used in the last section, since, in general, the thermodynamic description of the system is something which must be decided on before obtaining a set of hydrodynamic equations.

It is perhaps of interest to pause here to consider the relation between the thermodynamic argument given here and the argument given by Landau (presented in the preceding section). According to Landau, the observer moving with the superfluid
sees something quite similar to the motion of an ordinary fluid with a total energy $E_o$ (cf. equation (6)) per unit volume, momentum $\dot{J}_o = \xi_n \xi$ per unit volume and an entropy $\xi S$ per unit volume. Thus if we consider a volume $V$ moving with the superfluid, the total energy is (equations (51) and (54) evaluated in the rest frame of the superfluid)

$$E_o = VE_o = V \left\{ \frac{1}{2} \xi_n w^2 + \xi_n \xi_n + \xi_S \xi_S \right\} ,$$

the total momentum is $\dot{J}_o = V \xi_n \xi$, the total entropy is $S = \xi S V = \xi_n S_n V$, and the total mass is $M = V (\xi_n + \xi_S)$. On the basis of Landau's argument and the energy minimum principle of thermodynamics, we might expect the equilibrium to be determined by

$$\frac{dE_o}{\dot{J}_o, V, M, S} = 0 \quad (68)$$

and, in fact, (68) yields the condition (61) ($\Phi_n - \Phi_S = \frac{1}{2} w^2$).

We now consider the derivation of the hydrodynamic equations for reversible processes. The main principles to be used here are (i) analogy with ordinary hydrodynamics, and (ii) the assumption that there is no coupling between the fluids except the $\xi_n \leftrightarrow \xi_S$ transitions.

First of all, there will be two continuity equations which, however, must take into account the $\xi_n \leftrightarrow \xi_S$ transitions which conserve only the total mass. Thus if we let $\Gamma$ be the volume rate of conversion of superfluid into normal fluid, we may write the continuity equations as
\[ \frac{\partial \varepsilon_n}{\partial t} + \text{div} (\varepsilon_n v_n) = \Gamma, \]

and

\[ \frac{\partial \varepsilon_s}{\partial t} + \text{div} (\varepsilon_s v_s) = -\Gamma. \]  

(Again, \( \Gamma \) is not independently specified, but is determined by the flow conditions.)

To obtain the momentum equations, we assume that each fluid simply satisfies an Euler equation (with the normal fluid pressure \( p_n \) acting only on the normal fluid and the superfluid pressure \( p_s \) acting on the superfluid) with the momentum exchange due to the transitions \( \varepsilon_n \leftrightarrow \varepsilon_s \) included.

Since the transitions conserve total momentum, we may write the two momentum equations as

\[ \frac{\partial}{\partial t} (\varepsilon_n v_n) + \text{div} (\varepsilon_n v_n v_n) = -\nabla p_n + \mathcal{J}, \]

\[ \frac{\partial}{\partial t} (\varepsilon_s v_s) + \text{div} (\varepsilon_s v_s v_s) = -\nabla p_s - \mathcal{J}, \]

where \( \mathcal{J} \) represents the effect of transitions. \( \mathcal{J} \) may be expressed in terms of the transition rate \( \Gamma \) as follows: in a unit volume, there is superfluid momentum of amount \( \varepsilon_s v_s \); the transitions do not change \( v_s \), and the change in \( \varepsilon_s \) in time \( \partial t \) due to transitions is \( -\Gamma \partial t \); thus the rate of change of \( \varepsilon_s v_s \) due to transitions is \( -\Gamma v_s \), and, since the transitions conserve the total momentum, the rate of change of the
normal fluid momentum is

\[ \mathbf{j} = \nabla \cdot \mathbf{v}_s. \]  \hspace{1cm} (71)

By using (69) and (71), we may write (70) as

\[ \frac{\partial v_n}{\partial t} + v_n \cdot \nabla v_n = -\nabla p - \frac{\nabla \cdot w}{c_n} \]  \hspace{1cm} (72)

and

\[ \frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s = -\frac{\nabla p_s}{\xi_s} \]  \hspace{1cm} (73)

or, by (67),

\[ \frac{\partial v_n}{\partial t} + v_n \cdot \nabla v_n = -\frac{\nabla p_s}{\xi_s} - (1-x) s \nabla T - (1-x) \nabla \frac{1}{2} w^2 \]  \hspace{1cm} (74)

\[ \frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s = -\frac{\nabla p_s}{\xi_s} + s \nabla T + x \nabla \frac{1}{2} w^2 \]  \hspace{1cm} (75)

and these are the same as the equations originally obtained by Landau, without, however, the restriction \( \text{curl } v_s = 0 \).

Finally, we have the equation expressing the conservation of entropy, which we take as

\[ \frac{\partial}{\partial t} (\rho s) + \text{div} (\rho s v_n) = 0. \]  \hspace{1cm} (76)

It is easy to show that the above equations lead to the same equations for \( \rho \) and \( E \) as were obtained by means of the conservation laws. In terms of the partial pressures \( p_n, p_s \), the energy equation may be written as
\[ \frac{\partial}{\partial t} \left[ \frac{1}{2} p_n v_n^2 + p_n e_n + \frac{1}{2} \epsilon_s v_s^2 + \epsilon_s e_s \right] + \text{div} \left[ \left[ p_n + p_n e_n + \frac{1}{2} \epsilon_n v_n^2 \right] v_n \right] + \left[ p_s + \epsilon_s e_s + \frac{1}{2} \epsilon_s v_s^2 \right] v_s = 0, \]  

(77)

and the energy equations for each component read

\[ \frac{\partial}{\partial t} \left[ \frac{1}{2} p_n v_n^2 + p_n e_n \right] + \text{div} \left[ \left[ p_n + p_n e_n + \frac{1}{2} \epsilon_n v_n^2 \right] v_n \right] = \Gamma \left[ \Phi_n - \frac{1}{2} w^2 v_n + \frac{1}{2} v_s^2 \right], \]  

(78)

\[ \frac{\partial}{\partial t} \left[ \frac{1}{2} p_s v_s^2 + p_s e_s \right] + \text{div} \left[ \left[ p_s + \epsilon_s e_s + \frac{1}{2} p_s v_s^2 \right] v_s \right] = -\Gamma \left[ \Phi_s + \frac{1}{2} v_s^2 \right]. \]  

(79)

where the terms on the right-hand sides give the energy exchange rate due to transitions (as may be verified by a calculation similar to the one used to obtain an expression for \( \mathcal{J} \)).

Thus we see that from the present point of view, each fluid satisfies the perfect fluid equations of ordinary hydrodynamics when suitable provisions are made for the \( \xi_n \rightarrow \epsilon_s \) transitions. In this sense, then, Landau's equations--without the restriction \( \text{curl} \ v_s = 0 \) --are the "simplest" possible equations for the hydrodynamics of the two-fluid model.

This derivation shows that if one combines the laws of ordinary thermodynamics and hydrodynamics with a quite literal interpretation of the two-fluid model, the Landau equations are obtained in an unambiguous manner. Although the final equations are consistent with the condition \( \text{curl} \ v_s = 0 \), they do not require it and they admit more general solutions for which \( \text{curl} \ v_s \neq 0 \).
3. Variational principle

In attempting to derive the hydrodynamic equations describing reversible processes in helium II, one might expect some sort of variational principle to hold. The first attempt to obtain the equations for helium II by means of a variational principle was made by Tisza in 1947 [36], but Tisza's final equations were valid only in a linear approximation. In 1950, Zilsel [40] derived Landau's equations by means of a variational principle; this derivation (also given in London's book [29]) will be discussed in detail below. First, however, there are some general difficulties with the variational principle to be discussed.

These difficulties stem from the fact that the variational principles of mechanics apply to a Lagrangian description of the motion, whereas a Lagrangian description of the motion of helium II within the framework of the two-fluid model is manifestly impossible. One would still hope to get some sort of "Eulerian" variational principle for the two-fluid model, but there are difficulties with such a variational principle even in the case of ordinary hydrodynamics. Of course, one may use a Lagrangian description for discussing the motion of an ordinary fluid and obtain the hydrodynamic equations in a straightforward manner from a variational principle (Herivel, [16]). However, we wish to obtain a variational principle directly in terms of the Eulerian description of the motion, since this is the only sort of variational
principle which we may hope to generalize to include the two-fluid model. As a preliminary to the discussion of the variational principle for the two-fluid model, we consider the case of an ordinary fluid.

As a first approach, we consider the Lagrangian function

$$ L = \int_V d^3x \left\{ \frac{1}{2} \rho \nu^2 - \rho e \right\}, $$

(79)

where $\nu$ is the velocity, $\rho$ the density and $e(\rho)$ is the specific internal energy, $s$ being the specific entropy. The conditions that mass and entropy be conserved are to be incorporated by means of Lagrangian multipliers, so that the modified Lagrangian is given by

$$ L' = \int_V d^3x \left\{ \frac{1}{2} \rho \nu^2 - \rho e - \alpha \left[ \frac{\partial e}{\partial t} + \text{div} (\rho \nu) \right] - \beta \left[ \frac{\partial s}{\partial t} (\rho s) + \text{div} (\rho s \nu) \right] \right\}, $$

(80)

where $\alpha$, $\beta$ are the Lagrangian multipliers. The variational principle is then simply $\delta \int d\tau L' = 0$, with $\rho$, $s$ and $\nu$ varied independently. The three variational equations are

$$ \delta \nu: \quad \rho \left\{ \nu + \nu \alpha + s \nu \beta \right\} = 0, $$

(81)

$$ \delta \rho: \quad \frac{1}{2} \nu^2 - e - \rho + \frac{D\alpha}{Dt} + s \frac{D\beta}{Dt} = 0, $$

(82)

and $\delta s$: $\frac{D\beta}{Dt} = \tau$,

(83)

where $\rho$, $\tau$ are given by

$$ de = \frac{D\rho}{Dt} d\rho + T ds, $$

(84)

and $\frac{D}{Dt} = \nu \cdot \nabla + \frac{\partial}{\partial t}$. Elimination of $\alpha$ and $\beta$ leads to the usual
momentum equation, \( \rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v = -\nabla p \). However, \( \nabla \cdot v = 0 \) vanishes whenever \( \mathbf{s} \) is uniform in space. Thus this Eulerian variational principle (apparently due to Eckart [5] originally) yields the hydrodynamic equations, but also restricts the class of possible solutions. However, Lin [33] has shown how one may modify the Eulerian variational principle in such a way as to make it equivalent to the general hydrodynamic equations. According to Lin one must take into account the fact that the velocity field \( v \) is actually the (substantial) time derivative of a displacement field; in order to take this into account in the variational principle, one introduces the Lagrangian coordinates \( \mathbf{x}(\mathbf{X}, t) \) of the fluid particles and the constraints

\[
\frac{\partial \mathbf{X}}{\partial t} = 0. \tag{85}
\]

The constraints may be introduced by a Lagrange multiplier, \( \lambda \), and the Lagrangian function is then taken as

\[
L' = \int \left[ \frac{1}{2} \rho v^2 - \rho e - \alpha \left[ \frac{\partial p}{\partial t} + \text{div}(\rho v) \right] - \beta \left[ \frac{\partial}{\partial t}(\rho s) + \text{div}(\rho sv) \right] - \rho \mathbf{v} \cdot \frac{\partial \mathbf{X}}{\partial t} \right] \, dV. \tag{86}
\]

The variational equations are obtained by independently varying \( v, \mathbf{X}, p \) and \( s \); these equations can be written as

\[
\delta v: \quad v + \nabla \alpha + \mathbf{s} \nabla \beta - \mathbf{Y}_j \nabla X_j = 0, \tag{87}
\]

\[
\delta p: \quad \frac{1}{2} \rho v^2 - e - \frac{p}{\rho} + \frac{\partial \alpha}{\partial t} + \frac{\partial \beta}{\partial t} = 0, \tag{88}
\]

\[
\delta s: \quad \frac{\partial \beta}{\partial t} = T, \tag{89}
\]
and

\[ \delta x : \quad \frac{Dx}{Dt} = 0. \]

Elimination of the multipliers \( \alpha \), \( \beta \) and \( \gamma \) leads to the usual hydrodynamic equation, \( \rho \frac{DV}{Dt} = -\nabla P \), but this time \( \text{curl } \gamma \) does not necessarily vanish, even when \( \rho \) is uniform in space. Serrin [23] has completed the equivalence proof by showing that every solution of the hydrodynamic equations is also an extremal for the variational principle.

Although the above modification works for ordinary hydrodynamics (and in fact was used by Lin [24] to obtain the equations for helium II within the framework of his one-fluid model), it does not help in the search for a variational principle for the two-fluid model, since, in introducing the constraint (85), we have had to appeal directly to the Lagrangian nature of the system. Of course one can formally introduce constraints of the form (85) for each of the velocity fields, but the physical significance of such a device for the two-fluid model is not clear. Whitlock [23] has considered some extensions of Zilsel's work along these lines, and this work will be briefly discussed below.

We now consider Zilsel's derivation of the hydrodynamic equations for the two-fluid model. The starting point is the usual two-fluid model with velocities \( \mathbf{v}_n \), \( \mathbf{v}_s \), density \( \rho \), specific entropy \( s \) and normal fluid concentration \( x = \rho_n/\rho \). The laws of conservation of mass and entropy,
\[
\frac{\partial \rho}{\partial t} + \text{div}\{\rho [x v_n + (1-x) v_s]\} = 0,
\]
(91)

\[
\frac{\partial (\rho s)}{\partial t} + \text{div}\{\rho s v_n\} = 0,
\]
(92)

are to be treated as constraints in the variational principle. The Lagrangian density is taken as \( \frac{1}{2} \rho x v_n^2 + \frac{1}{2} \rho (1-x) v_s^2 - \rho e \), where \( e \) is the specific internal energy and is assumed to be a function of \( \rho, s \) and \( x \). Then the variational principle takes the form

\[
\delta \int_{t_0}^{t_f} \int \left[ \frac{1}{2} x v_n^2 + \frac{1}{2} \rho (1-x) v_s^2 - \rho e - \alpha \left( \frac{\partial \rho}{\partial t} + \text{div}\{\rho (x v_n + (1-x) v_s)\} \right) 
- \beta \left( \frac{\partial (\rho s)}{\partial t} + \text{div}\{\rho s v_n\} \right) \right] dt dv = 0,
\]
(93)

where \( \alpha \) and \( \beta \) are Lagrange multipliers. In the variation (93), the nine quantities \( \rho, s, x, v_n, v_s \) are varied independently although, as mentioned earlier, the number of independent quantities in the two-fluid model is eight; it turns out that the variational principle gives the thermodynamic equilibrium relation, as well as the dynamical equations. The variational equations are

\[
\delta \rho : \frac{1}{2} [x v_n^2 + (1-x) v_s^2] - e - \frac{\rho}{\rho} + \frac{\partial \alpha}{\partial t} + [x v_n + (1-x) v_s] \cdot \nabla \alpha = 0,
\]
(94)

\[
+ s \left( \frac{\partial \beta}{\partial t} + v_n \cdot \nabla \beta \right) = 0,
\]

\[
\delta s : \frac{\partial \beta}{\partial t} + v_n \cdot \nabla \beta = 0,
\]
(95)

\[
\delta x : \frac{1}{2} (v_n^2 - v_s^2) + (v_n - v_s) \cdot \nabla \alpha = \left( \frac{\partial \rho}{\partial x} \right)_{\rho, s, \gamma}
\]
(96)
\[ \delta v_n : \quad v_n + \nabla \alpha + \frac{s}{x} \nabla \beta = 0, \]  

and

\[ \delta v_s : \quad v_s + \nabla \alpha = 0, \]  

where

\[ T = \left( \frac{\partial e}{\partial s} \right) e_s x, \quad \frac{P}{e_2} = \left( \frac{\partial e}{\partial e} \right) e_s x. \]  

The elimination of the multipliers \( \alpha \) and \( \beta \) leads to the equations

\[ \frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s = -\frac{1}{e} \nabla p + s \nabla T + \frac{x}{2} \nabla w^2; \]  

\[ \frac{\partial v_n}{\partial t} + v_n \cdot \nabla v_n = -\frac{1}{e} \nabla p - \frac{1-x}{x} s \nabla T - \frac{1-x}{2} \nabla w^2 - \frac{w}{e x} \]  

(where \( \nabla = \frac{\partial e_n + \text{div} (e_n v_n)}{} \))

and

\[ \left( \frac{\partial e}{\partial x} \right) e_s s = \frac{1}{2} w^2. \]  

Equations (100) -(103) along with (91) and (92) are exactly the Landau equations discussed in the previous sections. However, the variational equations also imply the restrictive equations

\[ \text{curl } v_s = 0 \]  

\[ \text{(104)} \]
and \( \text{curl} \, v_S = \nabla S_n \times \nabla \beta \) \quad (S_n = S / x). \quad (105)

It is perhaps worth noting that if a superfluid entropy \( s_s \) had been included in the formulation, then the restrictive equations would have taken the form

\[
\text{curl} \, v_S = \nabla S_s \times \nabla \beta \quad (106)
\]

and

\[
\text{curl} \, v_n = \nabla S_n \times \nabla \beta, \quad (107)
\]

and these equations are very similar to the restrictive equations obtained from the unmodified variational principle for an ordinary fluid. Although there is no reason to believe that the superfluid entropy is not zero, the equations (106) and (107) lend support to the idea that the restrictive equations (104) and (105) stem from a defect in the variational principle and do not have physical content. (The fact that the variational principle yields equation (104) is often used as an argument in support of the physical statement that the superfluid cannot rotate; the restrictive equation (105), which also follows from the variational principle, is, however, usually ignored.)

In an attempt to remove the restrictions on the velocity fields, Whitlock [39] has discussed the effect of including constraints of the type (85) in Zilsel's variational approach. He found that if one introduces constraints of the type (85) for both the superfluid and normal fluid velocities, it is difficult to eliminate the Lagrange multipliers without going
to higher order equations of motion. He also found that if
one introduces a constraint only on the normal fluid velocity,
(i) the resulting equations of motion are the same as those
obtained above ((100) - (103)), (ii) there is no restrictive
equation of the type (105) on the normal fluid velocity, and
(iii) there is still the restriction that \( \text{curl} \mathbf{v}_n = 0 \); thus
in this way the Landau equations (and the constraint \( \text{curl} \mathbf{v}_n = 0 \))
are equivalent to a modified variational principle. However,
the physical significance of introducing the "Lagrangian co-
ordinates" of the normal fluid is not clear, both because of
the possibility of \( \mathcal{P}_n \leftrightarrow \mathcal{P}_s \) transitions (as Whitlock points out),
but, even more so, because of the impossibility of identifying
a given fluid element with the normal fluid or superfluid com-
ponent.

Although Lin was able to resolve the difficulties with
the Eulerian variational principle in ordinary hydrodynamics,
it is significant that his resolution involved a direct appeal
to the Lagrangian nature of the fluid. The two-fluid model,
however, defies a Lagrangian description, and a complete reso-
lution of the difficulties with the two-fluid variational
principle is yet to be given. The difficulties in the two-
fluid model associated with the lack of a Lagrangian descrip-
tion are fundamental ones, and, to quote London ([29],
p. 127), "they appear so intrinsic that we might question
whether the two-fluid concept is actually compatible with the
principles of classical particle mechanics." One way around
these difficulties is to abandon the two-fluid model. In the
next section we give a detailed discussion of the hydrodynamics of helium II on the basis of Lin's one-fluid model[24].
B. One-Fluid Model

1. Introduction

Although the two-fluid model has served as the starting point for most discussions of the hydrodynamics of helium II, Lin([26,25,24]) has shown that a consistent hydrodynamic theory for helium II may be developed in a natural way by a suitable generalization of the concepts of ordinary hydrodynamics. This section will be devoted to a detailed discussion of Lin's one-fluid theory for reversible processes.

Lin's theory is a direct generalization of the ordinary hydrodynamics of a single fluid. The starting point for the generalization is the experimental observation that helium II at rest can transmit heat in a reversible manner. This fact is accounted for in Lin's theory by introducing, as a new thermodynamic variable, the entropy flux $\bar{S}$ relative to the fluid; then one may define a velocity of internal convection $\bar{c}$ by the relation

$$\bar{S} = \rho \bar{c}$$

where $\rho$ is the density and $\bar{s}$ the specific entropy. The other macroscopic variable needed for the description of the helium II motion is the macroscopic velocity $\bar{v}$. The central problem, then, is to determine the hydrodynamic equations governing the behavior of the quantities $\rho$, $\bar{s}$, $\bar{v}$, and $\bar{c}$. Two equations may be immediately written down describing the laws of conservation of mass and conservation of entropy;
these are

\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0, \]  

(109)

and

\[ \frac{\partial}{\partial t}(\rho \mathbf{s}) + \text{div}\{\rho \mathbf{s}(\mathbf{v} + \mathbf{c})\} = 0. \]  

(110)

Two further (vector) equations are needed -- one each for \( \mathbf{v} \) and \( \mathbf{c} \). The derivation of these equations will be given in section II-B-2 (variational principle) and section II-B-3 (conservation laws). As in the case of the two-fluid model, some discussion of the thermodynamics of the system is a necessary preliminary to the derivation of the hydrodynamic equations.

It is assumed that there exists an internal energy function \( e \), such that the total energy per unit volume is given by

\[ E = \frac{1}{2} \rho \mathbf{v}^2 + \rho e. \]  

(111)

The function \( e \) will depend on the thermodynamic variable \( s \), as well as the quantities \( \rho \) and \( e \); thus in general,

\[ e = e(\rho, s, s), \quad s = \frac{1}{2} c^2. \]  

(112)

Equation (112) does not give a complete specification of the thermodynamics of the system, however, because one must know the proper extensive variable associated with the quantity \( c \), in order to identify the pressure and temperature with the derivatives of \( e \) with respect to \( \rho \) and \( s \). This is a
difficult point and one which cannot be resolved without some further information on the physical role of the quantity $c$; however, the derivation of the hydrodynamic equations in section II-B-2 gives rise to a situation in which there is a "natural simplest" choice. Further discussion of this point is then deferred to the next section.
2. Variational principle

For reversible processes, one expects that the equations may be obtained from a variational principle, and, as a natural generalization of ordinary hydrodynamics, it is assumed that the equations for both \( \mathbf{v} \) and \( \mathbf{e} \) may be obtained from a variational principle. In writing down a Lagrangian density for the system, one must know how to split the energy (111) into a potential part and a kinetic part. For an ordinary fluid, the correct variational principle is obtained by including the thermodynamic internal energy with the potential energy of the system; in the present case, however, one must allow for the possibility that some of the thermodynamic energy \( \mathbf{e} \) (112) is actually kinetic in nature. Thus we write the energy \( \mathbf{e} \) as

\[
\mathbf{e} = \mathbf{e}_p + \mathbf{e}_k \tag{113}
\]

where \( \mathbf{e}_p \) is the potential part, \( \mathbf{e}_k \) the kinetic part. Then it is to be expected that the Lagrangian density will be given by

\[
\mathcal{L} = \rho \left[ \frac{1}{2} \mathbf{v}^2 - \mathbf{e}_L \right], \tag{114}
\]

where

\[
\mathbf{e}_L = \mathbf{e}_p - \mathbf{e}_k. \tag{115}
\]

At this point, \( \mathbf{e}_p \) and \( \mathbf{e}_k \) are not known; however, we shall see that the correct splitting (113) may be deduced from the equations of motion.
The variation is to be carried out with the equations (109) and (110) incorporated as constraints; in accordance with the discussion of II-A-3, we introduce the Lagrangian coordinates \( \mathbf{X} \), and the constraint

\[
\frac{\mathbf{D}\mathbf{X}}{\mathbf{D}t} = 0. \tag{116}
\]

The variational principle is then

\[
\delta \int_{t_i}^{t_f} \int_V \left\{ \rho \left( \frac{1}{2} \mathbf{v}^2 - e_\Lambda \right) - \alpha \left[ \mathbf{\nabla} \cdot \mathbf{\epsilon} + \text{div}(\mathbf{\epsilon} \mathbf{v}) \right] - \beta \left[ \mathbf{\nabla} \cdot (\mathbf{s} \mathbf{\epsilon}) + \text{div}(\mathbf{s} \mathbf{\epsilon} \mathbf{v} \mathbf{\sigma}) \right] \right\} dt
\]

\[- \lambda \mathbf{\epsilon} \cdot \mathbf{\frac{\mathbf{D}\mathbf{X}}{\mathbf{D}t}} = 0 \tag{117}\]

where \( \mathbf{\epsilon} \), \( \mathbf{s} \), \( \mathbf{v} \), \( \mathbf{\epsilon} \), \( \mathbf{X} \) are to be varied independently, and \( \alpha \), \( \beta \), \( \lambda \) are Lagrangian multipliers. The variational equations are

\[
\delta \rho: \quad - e_\Lambda - \rho_L + \frac{1}{\lambda} \mathbf{v}^2 + \mathbf{\nabla} \alpha + \mathbf{s} \mathbf{D} \beta + \mathbf{s} \mathbf{\epsilon} \cdot \mathbf{\nabla} \beta = 0, \tag{118}\]

\[
\delta \mathbf{s}: \quad \frac{\mathbf{D} \beta}{\mathbf{D}t} + \mathbf{\epsilon} \cdot \mathbf{\nabla} \beta = \mathbf{T}_L, \tag{119}\]

\[
\delta \mathbf{v}: \quad \mathbf{v} + \mathbf{\nabla} \alpha + \mathbf{s} \mathbf{\nabla} \beta = \sum_{i=1}^{3} \lambda_i \mathbf{\nabla} \mathbf{X}_i, \tag{120}\]

\[
\delta \mathbf{\epsilon}: \quad \mathbf{s} \mathbf{\nabla} \beta = \mathbf{z} \mathbf{\epsilon}, \tag{121}\]

and
\[ \delta X: \quad \frac{D \lambda}{D t} = 0, \quad (122) \]

where we have introduced the derivatives of \( \mathbf{e}_L \), namely,

\[ d e_L = T_L dS + \frac{P_L}{\mathbf{e}_L} d\rho + z dS. \quad (123) \]

The final hydrodynamic equations are obtained by eliminating the Lagrange multipliers, and they may be written as

\[ \frac{\partial e}{\partial t} + \text{div}(\mathbf{e} \mathbf{v}) = 0, \quad (124) \]

\[ \frac{\partial}{\partial t} (\mathbf{e} \mathbf{s}) + \text{div}\{\mathbf{e} s (\mathbf{v} + \mathbf{z})\} = 0, \quad (125) \]

\[ \mathbf{e} \frac{\partial \mathbf{v}}{\partial t} = \text{div} \mathbf{z}, \quad \mathbf{z} = -P_L \mathbf{I} + z \mathbf{e} \mathbf{c} \mathbf{c}, \quad (126) \]

and

\[ \frac{\partial}{\partial t} \left( \frac{\mathbf{z} \mathbf{c}}{\mathbf{s}} \right) + \nabla \left\{ \frac{\mathbf{z} \mathbf{c}}{\mathbf{s}} \cdot (\mathbf{v} + \mathbf{z}) - T_L \right\} = 0, \quad (127) \]

or

\[ \frac{\partial}{\partial t} (\rho z c_i) + \frac{\partial}{\partial x_j} \left\{ (v_i + c_j) \rho z c_i \right\} = \rho s e \frac{\partial T_L}{\partial x_i} - \rho z c_i \frac{\partial}{\partial x_i} (v_i + c_j). \]

Thus the reversible heat transfer gives rise to a momentum transfer, as is evident from (126). We may also write (126) in a form which expresses the law of conservation of momentum, namely,

\[ \frac{\partial}{\partial t} (\mathbf{e} \mathbf{v}) + \text{div} \mathbf{w} = 0, \quad (128) \]
where the momentum flux tensor $\Pi$ is given by

$$\Pi_{ij} = e v_i v_j + p_\perp \delta_{ij} - \rho z c_i c_j. \quad (129)$$

There are several points to discuss in completing the derivation. We first consider the problem of determining the energy parts $e_p$ and $e_k$. The basic principle to be used is the fact that, in the perfect fluid theory, the hydrodynamic equations (124) - (127) must imply the conservation of total energy. The energy density is $\rho = \frac{1}{2} e v^2 + \rho e$, and the energy equation must have the form

$$\frac{\partial}{\partial t} \left[ \frac{1}{2} e v^2 + \rho e \right] + \text{div} Q = 0, \quad (130)$$

where $Q$ is the (as yet undetermined) energy flux vector. Although $Q$ is unknown, we may gain some information about the form of $Q$ from the Galilean transformation formulae. If $Q_0$ denotes $Q$ as measured in the rest frame of the fluid, then $Q$ and $Q_0$ are related by

$$Q = E v + (\Pi_0 \cdot v) + Q_0. \quad (131)$$

where $\Pi_0$ is the momentum flux tensor as measured in the fluid rest frame - i.e., (from (124))

$$\Pi^\perp_{ij} = p_\perp \delta_{ij} - \rho z c_i c_j; \quad (132)$$

thus

$$Q = E v + p_\perp v - \rho z c (c \cdot v) + Q_0.$$
Since we are dealing with a perfect fluid theory, it is plausible to assume that $Q_0$ does not depend on any of the spatial derivatives of the quantities $\rho$, $s$, $v$ and $c$; since $Q_0$ is a Galilean invariant, it cannot depend on $v$, so that we have

$$Q_0 = \lambda c,$$  \hspace{1cm} (133)

where, in general, $\lambda = \lambda(\rho, s, \zeta)$. In the calculations, to follow, it is convenient to replace the unknown scalar $\lambda$ by another unknown scalar function $\lambda'$, given by

$$\lambda = \rho s T - \rho z c^2 + \lambda'.$$ \hspace{1cm} (134)

Then the energy flux is given by

$$Q = (\frac{1}{2}\rho v^2 + \rho \varepsilon + p_L)v - \rho z c (\varepsilon \cdot v) + \rho s T c - \rho z c^2 c + \lambda' c.$$ \hspace{1cm} (135)

The calculations proceed as follows: from the hydrodynamic equations (124) -- (127), we may calculate $\frac{\partial E}{\partial t}$ directly in terms of $\rho$, $s$, $v$, $c$ and their spatial derivatives; if the result of this calculation and the expression (135) for $Q$ are substituted into the energy equation (130), the result is

$$\rho \frac{\partial \varepsilon'}{\partial t} + \text{div } \lambda' c = 0,$$ \hspace{1cm} (136)

where

$$\varepsilon' = \varepsilon - c_L + 2 z s.$$ \hspace{1cm} (137)
In the calculation of \( \rho \frac{D e'}{D t} \), it is convenient to regard \( e' \) as a function of \( \rho \), \( s \) and \( \kappa = \frac{1}{2} (\frac{Z}{\varepsilon})^2 \); then

\[
\rho \frac{D e'}{D t} = \rho \frac{D e'}{D t} + \rho \frac{D e'}{D s} \frac{D s}{D t} + \rho \frac{D e'}{D \kappa} \frac{D \kappa}{D t} \left(\frac{Z}{\varepsilon}\right),
\]

and we may use the hydrodynamic equations (124) - (127) and equation (130) to obtain

\[
-\{\delta_{ij} \frac{D e'}{D \rho} + \rho \frac{D e'}{D \kappa} \frac{Z c_i c_j}{\varepsilon} \} \frac{D v_i}{D x_j} - \frac{D e'}{D s} \frac{D (\rho s c_j)}{D x_j} - \rho \frac{D e'}{D \kappa} \frac{Z c_i}{\varepsilon} \frac{D (Z c_j^2 - T_j)}{D x_i}
+ \frac{D}{D x_j} (\lambda' c_j) = 0. \tag{138}
\]

Since no time derivatives appear in (138), it must be an identity; in particular the coefficient of \( \frac{D v_i}{D x_j} \) must vanish, so that

\[
\delta_{ij} \frac{D e'}{D \rho} + \rho \frac{D e'}{D \kappa} \frac{Z c_i c_j}{\varepsilon} \frac{Z c_j}{\varepsilon} \equiv 0. \tag{139}
\]

Equation (139) must hold for all values of \( \varepsilon \), and it is easy to show that

\[
\frac{D e'}{D \rho} = 0, \quad \frac{D e'}{D \kappa} = 0
\]

so that \( e' = e'(s) \). Then (139) becomes

\[
- \frac{D e'}{D s} \text{div}(\rho s c) + \text{div}(\lambda' c) \equiv 0. \tag{140}
\]

In (140), we may regard \( \lambda' \) as a function of \( \rho \), \( s \) and \( \frac{1}{2} \varepsilon^2 = \varepsilon \).

Then (140) becomes
\[
\left\{ c_i c_j \frac{\partial^2 \lambda'}{\partial s^2} + \lambda' \delta_{ij} - p s \frac{\partial e'}{\partial s} \delta_{ij} \right\} \frac{\partial c_i}{\partial x_j} + \left\{ c_i \frac{\partial \lambda'}{\partial s} - c_i e' \frac{\partial e'}{\partial s} \right\} \frac{\partial s}{\partial x_i} \equiv 0,
\]

and from here it is easy to show that

\[
\frac{d^2 e'}{ds^2} = 0, \quad (141)
\]

and

\[
\lambda' = p s \frac{de'}{ds}. \quad (142)
\]

Thus the most general form for \( e' \) allowed is \( e' = \gamma_1 s + \gamma_2 \), where \( \gamma_1, \gamma_2 \) are constants (not depending on any of the thermodynamic or flow variables). Then from (137), we obtain

\[
e_L = e + 2 zs - e' = e + 2 zs - \gamma_1 s - \gamma_2.
\]

It is clear that the terms \( \gamma_1 s + \gamma_2 \) contribute nothing to the variational equations, because the integral of \( p \gamma_1 s \) over the volume \( V \) is proportional to the total entropy and the integral of \( p \gamma_2 \) is proportional to the total mass, both of which are conserved. Thus we may take \( \gamma_1 = \gamma_2 = 0 \) without affecting the variational principle; then we have the final results

\[
e_L = e + 2 zs, \quad (143)
\]

and

\[
e_k = -zs, \quad e_p = e + zs.
\]
We may also note that a definite expression for the energy flux vector has been obtained from this calculation, namely

\[ Q = \left( \frac{1}{2} \rho v^2 + \rho e + p_L \right) v - \rho z \xi (\xi \cdot v) + \rho s T_l - \rho z c^2 \xi. \]  (144)

As mentioned in the preceding section, the theory is not really complete until we have identified the extensive variable associated with \( \xi \), as we may not obtain the pressure and temperature in terms of the derivatives of the energy function until this third extensive variable is known. At present, there seem to be no simple arguments within the framework of the one-fluid theory which would furnish a definite answer to this question. Thus in the absence of more definite information, we simply assume (on the basis of "maximum simplicity" of the hydrodynamic equations) that the quantities \( p_L \) and \( T_l \) may be identified with the thermodynamic pressure and temperature; that is,

\[ p = p_L, \quad T = T_l. \]  (145)

Then the differentials of the thermodynamic quantities \( \xi \), \( \xi_L \) and \( \xi_P \) become

\[ d \xi_L = \frac{p}{\xi^2} d \rho + T d s + z d s, \]  (146)

\[ d \xi = \frac{p}{\xi^2} d \rho + T d s - \xi \cdot d z \xi, \]  (147)

and

\[ d \xi_P = \frac{p}{\xi^2} d \rho + T d s - s d z. \]  (148)
In section II-B-4, where a discussion of the relation of the present theory with the two-fluid theory is given, it will be shown that the quantity \(\mathbf{z}c\) has a definite interpretation as an extensive variable (per unit mass) within the framework of the two-fluid theory.

With this identification of \(\mathbf{z}c\) as the third extensive variable, it is possible to give a physical interpretation to the terms in the energy flux \(\mathbf{Q}\). By using (130), (144) and (147) we may write the energy equation (130) in the form

\[
\frac{\partial}{\partial t}\left\{ \frac{1}{2} \rho v^2 + \rho e \right\} + \text{div} \left\{ \frac{1}{2} \rho v^2 v + \rho e v + \rho \left( \frac{\partial e}{\partial c} \right)_{s,c} \mathbf{z}c (v+c-v) + \rho \left( \frac{\partial e}{\partial z_c} \right)_{s,c} \mathbf{z}c (v+c-v) \right\} = \text{div} (\mathbf{v} \cdot \mathbf{\Xi}),
\]

(149)

(Where \(\mathbf{\Xi} = -\mathbf{p} \frac{\mathbf{I}}{s} + \rho \mathbf{z}c\)).

The term on the right gives the rate of working of the stresses; in the divergence on the left, we have the convection of kinetic energy, \(\frac{1}{2} \rho v^2 v\), and convection of internal energy, \(\rho e v\); the last two terms are corrections to the convection of internal energy, these terms occurring as a consequence of the fact that \(\mathbf{s}\) and \(\mathbf{z}c\) are convected with velocity \(v+c\), rather than \(v\).

At a stationary solid wall, the mass, entropy and energy fluxes must vanish (assuming there is no energy input to the fluid through the wall), so that

\[
\mathbf{v} \cdot \mathbf{n} = 0 ,
\]

\[
\mathbf{c} \cdot \mathbf{n} = 0 .
\]

(150)
(As mentioned before, it is necessary to consider irreversible processes in the discussion of the boundary conditions when there is a net energy flux between the wall and the fluid).

Lin [24] has presented some simple examples which show that the present theory predicts thermal waves (second sound) as well as ordinary sound waves.

There is one rather serious difficulty with the present derivation which should be discussed, and that is the fact that the variational equation (121) predicts that

$$\text{curl } \mathbf{c} = \nabla \left( \frac{s}{z} \right) \times \nabla \beta,$$  \hspace{1cm} (151)

or, alternatively,

$$\text{curl} \left( \frac{z \mathbf{c}}{s} \right) = 0,$$

so that \text{curl } \mathbf{c} vanishes whenever \( s/z \) is uniform in space. At this stage, one could accept (151) as having physical significance; however, there are several reasons for regarding (151) as an unphysical restrictive equation. First, equation (151) is very reminiscent of the restrictive equations obtained from the unmodified variational principle for an ordinary fluid and from Zilsel's variational principle for the two-fluid model, and, in these cases, at least some of the restrictive equations are due to defects in the variational principles. Second, we may note that a restrictive equation of the form (151) always arises whenever the Lagrangian density depends on a vector \( \mathbf{c} \) only through a term of the type \( \mathbf{e}_L(e,s,\frac{1}{2} \mathbf{c}^2) \) and a constraint term of the type \( \beta \text{ div}(\mathbf{e}\mathbf{c}) \). Finally we may note that, in
terms of the two-fluid model (cf. II-B-4), equation (151) predicts that \( \text{curl} (\mathbf{v}_n - \mathbf{v}_s) = \mathbf{0} \) whenever \( \mathbf{s} \) and \( x = \frac{\mathbf{e}_n}{\mathbf{e}} \) are uniform in space, and it is known that this equation is too restrictive to be in accord with some of the experimental results.

One might hope to remove the restriction (151) by modifying the variational principle in some way. However, the Lagrangian nature of the system has already been taken into account by means of the constraints (116) expressing the "conservation of particle identity", so it is not obvious as to what sort of modifications one should try. Formally, one can introduce "Lagrangian coordinates" for the velocity \( \mathbf{c} \), and constraints similar to (116); the physical significance of such a device is not clear, however, and the resulting equations for \( \mathbf{v} \) and \( \mathbf{c} \) (after elimination of the Lagrange multipliers) are of higher order than one would expect.

There is one important difference between the present case and the case of an ordinary fluid which may be a clue to the source of the difficulty. This is the fact that, for an ordinary fluid, the (local) conservation of entropy is equivalent to the statement that the entropy of each fluid particle stays constant, whereas in the one-fluid model for helium II, the entropy of each fluid particle does not stay constant (in general), even though the entropy is conserved. Thus in the present case, there is energy transfer of an essentially thermal nature (even though the transfer process is thermodynamically reversible), so that it is not entirely
clear that one can expect a variational principle of the form
\[ \delta \int dt \left( T - V \right) = 0 \]
to hold.

Since the variational principle is being used here to deduce the equations (and not to give an alternative derivation of the equations which are already known), it would seem to be important to resolve this difficulty; however, this has not been done as yet. In the absence of a more satisfactory derivation of the equations, we simply take the equations of motion to be those given by the above variational principle, without, however, retaining the restrictive equation (151). This procedure is somewhat arbitrary, since the various forms of the equations for \( \mathbf{a} \) (cf. equation (127)) are no longer equivalent when the restrictive equation (151) is dropped.

Of all the possible forms for the equation for \( \mathbf{a} \), one can perhaps justify a preference for the second of equations (127), since this is an equation for the time rate of change of the extensive quantity (per unit volume) \( \rho \mathbf{a} \). Thus we tentatively take the complete set of hydrodynamic equations in the following form:

\[
\frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{v}) = 0,
\]

\[
\frac{\partial}{\partial t} (\rho s) + \text{div} \{ \rho s (\mathbf{v} + \mathbf{a}) \} = 0, \tag{152}
\]

\[
\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} (\rho v_i v_j) = \frac{\partial}{\partial x_j} \left( -\rho \delta_{ij} + \frac{\partial}{\partial x_j} \left( \mathbf{a} \mathbf{a}_j \right) \right),
\]

and
\[ \frac{\partial}{\partial t}(\rho z c_i) + \frac{\partial}{\partial x_j}(\nu_j + c_j) \rho z c_i = \rho \rho z c_i \frac{\partial T}{\partial x_i} - \rho z c_j \frac{\partial}{\partial x_i} (\nu_j + c_j). \]

The thermodynamic relation (147) is the same as before, and the energy equation (which follows from (152)) is still given by (149).

Because of the difficulties in the derivation leading to (152), it would be very desirable to obtain the equations by another method. In the next section, we examine the possibility of deriving the equations from the conservation laws and the Galilean transformation formulae. It will be shown there that, although the equations (152) are consistent with the conservation laws and the Galilean transformation formulae, one cannot deduce a unique set of equations from that method.
3. Conservation laws

Landau's work (discussed in II-A-1) has shown that the imposition of the conservation laws plus the requirements of Galilean invariance are sufficient to determine a unique set of hydrodynamic equations (for reversible processes) for the two-fluid model, provided that it is assumed that always \( \text{curl} \, \mathbf{v}_s = 0 \). It was also shown in II-A-1 that it is no longer possible to deduce a unique set of hydrodynamic equations when the requirement \( \text{curl} \, \mathbf{v}_s = 0 \), is dropped. Thus it is no surprise that when the method is applied to Lin's one-fluid model, it does not lead to a unique set of hydrodynamic equations. The method does, however, limit the possible form of the hydrodynamic equations, and in particular, the equations (152) are consistent with the restriction imposed by the conservation laws and the requirements of Galilean invariance.

Since the method does not lead to a definite set of hydrodynamic equations, it is of limited interest, and the calculations will be described only very briefly here.

It is assumed that the macroscopic state is characterized by the values of \( \rho, \, s, \, \gamma \) and \( \mathbf{c} \); the mass flux is taken to be \( \rho \mathbf{v} \), the entropy flux to be \( \rho s (\gamma + \mathbf{c}) \). Then the laws of conservation of mass and entropy are the same as before,

\[
\frac{\partial \rho}{\partial t} + \text{div} (\rho \mathbf{v}) = 0, \quad (153)
\]

and

\[
\frac{\partial (\rho s)}{\partial t} + \text{div} \{ \rho s (\gamma + \mathbf{c}) \} = 0. \quad (154)
\]
The conservation of momentum is expressed by an equation of the form

\[ \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\tau_{ij}) = 0, \] (155)

where the momentum flux tensor \( \tau_{ij} \) is as yet undetermined. The total energy per unit volume is taken to be

\[ E = \frac{1}{2} \rho v^2 + \rho e, \] (156)

where \( e = e(\rho, s, z) \) is the specific internal energy, and

\[ de = \frac{P}{\rho^2} d\rho + T dS - c \cdot dz \] (157)

defines the pressure \( \rho \) and the temperature \( T \). (The thermodynamic description (157), obtained in the preceding section, is assumed to be valid here.) Then the conservation of energy is expressed by an equation of the form

\[ \frac{\partial E}{\partial t} + \text{div} \mathbf{Q} = 0, \] (158)

where the energy flux vector \( \mathbf{Q} \) is as yet undetermined. Finally, there will be an equation for the convection velocity \( \mathbf{c} \), and it is convenient to write this equation in the form

\[ \frac{\partial}{\partial t}(\rho z c_i) + \frac{\partial}{\partial x_j} \left[ (v_j + c_j) \rho z c_i \right] = F_i, \] (159)

where \( F_i \) is as yet undetermined. Although the quantities \( \tau_{ij} \), \( \mathbf{Q} \) and \( F \) are not known, we may use the Galilean
transformation formulae to express $\mathcal{Q}$ and $\pi_{ij}$ in terms of their values, $Q_o$ and $\pi_{ij}^o$, in the rest frame of the fluid, and we may also deduce that $F$ must be a Galilean invariant. Thus

$$\pi_{ij} = \mathcal{Q} v_i v_j + \pi_{ij}^o,$$

and

$$\mathcal{Q} = E v + (\pi^o \cdot v) + Q_o.$$

For convenience in the calculations to follow, we introduce new unknown quantities $\pi_{ij}'$, $Q'$ and $E'$ by the equations

$$\pi_{ij}' = \rho \delta_{ij} - \rho z c_i c_j + \pi_{ij}^o,$$

$$Q' = \rho s T - \rho z c^2 + Q',$$

and

$$F_i = \rho s \frac{\partial T}{\partial x_i} - \rho z c_j \frac{\partial}{\partial x_i} (v_j + c_j) + F_i'.$$

The quantities $\pi_{ij}'$, $Q'$ and $E'$ are all Galilean invariants, and the problem of determining the hydrodynamic equations is reduced to the problem of determining $\pi_{ij}'$, $Q'$ and $E'$.

The procedure is the same as before; we have 8 independent quantities ($\rho$, $s$, $v$, $c$) and 9 (scalar) equations for these quantities. We may obtain an equation which the quantities $\pi_{ij}'$, $Q'$ and $E'$ must satisfy in the following manner: we first calculate $\frac{\partial E}{\partial t}$ in terms of the macroscopic variables (and their spatial derivatives) directly from the hydrodynamic equations; then this expression for $\frac{\partial E}{\partial t}$ and the above expression for $\mathcal{Q}$ are substituted into the energy equation (158). The final result of the calculation is the
\[ 0 = \tau_{ij} \frac{\partial v_i}{\partial x_j} + \text{div } \mathbf{Q}' - \mathbf{c} \cdot \mathbf{F}'. \]  

(160)

Since this equation contains no time derivatives, it must be an identity, and it thus restricts the possible choices of \( \tau_{ij}, \mathbf{Q}', \text{ and } \mathbf{F}' \). (We may note that the equations (152) of the preceding section correspond to the choice \( \tau_{ij} \equiv 0, \mathbf{Q}' \equiv 0 \) and \( \mathbf{F}' \equiv 0 \).) It is easy to show from (160) that the most general expression for \( \mathbf{F}' \) (for arbitrary \( \tau_{ij} \) and \( \mathbf{Q}' \)) is given by

\[ \mathbf{F}' = \frac{1}{c^2} \mathbf{c} \left\{ \tau_{ij} \frac{\partial v_i}{\partial x_j} + \text{div } \mathbf{Q}' \right\} + \mathbf{\alpha} \times \mathbf{c}, \]  

(161)

where \( \mathbf{\alpha} \) is an arbitrary (Galilean invariant) vector. The fluxes \( \tau_{ij} \) and \( \mathbf{Q}' \) are Galilean invariant; thus if one makes the plausible assumption that these fluxes do not depend on the gradients of the macroscopic quantities, then one can show that the most general expressions for \( \tau_{ij} \) and \( \mathbf{Q}' \) are

\[ \tau_{ij}' = \lambda_1 c_i c_j + \lambda_2 c^2 \delta_{ij} \]

and

\[ \mathbf{Q}' = \lambda_3 \mathbf{c} \]

where \( \lambda_1, \lambda_2, \lambda_3 \) are scalar functions of \( \mathbf{c}, \mathbf{\alpha} \) and \( \frac{1}{2} c^2 \). Even with this simplification, there are still 3 arbitrary scalar functions \( (\lambda_1, \lambda_2, \lambda_3) \) and an arbitrary (Galilean invariant) vector \( \mathbf{\alpha} \). Thus this method by no means yields a definite set of hydrodynamic equations.
One can simplify the above results a little by giving a physical argument to determine the energy flux vector. On the basis of the discussion in the preceding section, we expect that the convective terms in the energy flux are of the form
\[ \frac{1}{\alpha} \rho \nu^2 \nu + \varepsilon \varepsilon \nu + \varepsilon \varepsilon \varepsilon = - \varepsilon \varepsilon c^2 c, \]
and that the terms accounting for the rate of working of the fluid stresses are of the form \[(\Pi^0 \cdot \nu),\]
so that the total energy flux is then
\[ Q = E \nu + \Pi^0 \cdot \nu + \varepsilon \varepsilon \varepsilon - \varepsilon \varepsilon c^2 c. \]
This corresponds to \( Q' = 0 \), so that (161) now reads
\[ F' = \frac{1}{c^2} c \tau_{ij} \frac{\partial \nu_i}{\partial x_j} + a \times c. \] (162)

There does not seem to be any simple argument which would allow an unambiguous determination of the stress tensor (and consequently of \( \pi_{ij} \)), but even if we accept the form of the stress tensor as given by the variational principle (corresponding to \( \pi_{ij} = 0 \)), we still have in \( F \) an arbitrary vector \((a \times c)\) perpendicular to \( c \). If we agree to accept the momentum flux tensor and the energy flux vector as obtained from the variational principle (i.e., we take \( Q' = 0, \pi_{ij} = 0 \)), then the most general set of equations consistent with the
the conservation laws and the requirements of Galilean invariance is

\[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0, \]

\[ \frac{\partial (\rho s)}{\partial t} + \text{div}\{\rho s (\mathbf{v} + \mathbf{c})\} = 0, \]

\[ \frac{\partial (\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \frac{\partial}{\partial x_j}(-p \delta_{ij} + \rho z c_i c_j), \]

and

\[ \frac{\partial (\rho z c_i)}{\partial t} + \frac{\partial}{\partial x_j}\left\{ (v_j + c_j) \rho z c_i \right\} = \rho s \frac{\partial T}{\partial x_i} - \rho z c_j \frac{\partial}{\partial x_i} (v_j + c_j) + F'_i, \]

where \( F'_i \) is an arbitrary Galilean invariant vector perpendicular to \( \mathbf{c} \). These are the same as the equations (152), except for the term \( F' \) in the equation for \( \mathbf{c} \).

In summary: the method yields a restrictive equation which the momentum flux tensor, the energy flux vector and the "force" \( F \) must satisfy, but does not yield a unique set of hydrodynamic equations. The equations (152) obtained from the variational principle are consistent with this restrictive equation.
4. Relation with the two-fluid theory

As Lin ([25, 24]) has shown, the hydrodynamic equations of his one-fluid theory are essentially equivalent to Landau's equations for the two-fluid model. In fact, the equations for the two-fluid model may be obtained from the equations of the one-fluid model by means of a simple mathematical transformation.

In the one-fluid model, the state of the fluid is completely described by the quantities $\rho$, $s$, $v$ and $\zeta$ (and the internal energy function); the basic variables of the two-fluid model may be taken as $\rho$, $s$, $v_n$, $v_s$ and $x = \frac{\rho_n}{\rho}$ (where $x$ is not independent, but may be regarded as a function of the other two-fluid variables). Thus if we specify $v_n$ and $v_s$ in terms of the one-fluid variables, the equations (152) (or (163)) of the one-fluid model will yield a set of equations for $\rho$, $s$, $v_n$, $v_s$ which are then to be compared with Landau's equations. In both models, we have definite expressions for the densities and fluxes of mass, entropy, momentum and energy; it turns out that the corresponding quantities in the two models are all equal, if we have

$$v_n = v + \zeta,$$

$$v_s = v + z \zeta,$$

$$\frac{\rho_n}{\rho} = x = \frac{z}{z-1} \quad (\text{or} \quad z = \frac{-x}{1-x}).$$
Of course, if we regard the one-fluid model as basic, then (164) may be regarded as simply a mathematical definition of the quantities \( \nu_n \), \( \nu_s \) and \( x \). Then we may further define
\[
\eta_n = \frac{x \rho_0}{2 - 1}, \quad \eta_s = (1 - x) \rho_0 = \frac{\rho_0}{1 - \frac{x}{2}}.
\]
By using the relations (164), we may (after some calculations) write the equations (163) in the following form:
\[
\frac{\partial \eta}{\partial t} + \text{div} \{ \eta_n \nu_n + \eta_s \nu_s \} = 0,
\]
\[
\frac{\partial \eta_s}{\partial t} + \text{div} \{ \eta_s \nu_n \} = 0,
\]
and
\[
\begin{aligned}
\frac{\partial}{\partial t} (\eta_n \nu_i + \eta_s \nu_{si}) + \frac{\partial}{\partial x_j} (\eta_n \nu_i \nu_j + \eta_s \nu_{si} \nu_{sj}) &= -\frac{\partial P}{\partial x_i}, \\
\frac{\partial \nu_s}{\partial t} + \nu_s \cdot \nabla \nu_s &= -\frac{1}{\eta} \nabla p + s \nabla T + \frac{x}{2} \nabla w^2 + x \nabla x (\text{curl} \nu_s) \\
&+ \frac{F'}{\eta},
\end{aligned}
\]
( where \( w = \nu_n - \nu_s \))
or
\[
\begin{aligned}
\frac{\partial \nu_n}{\partial t} + \nu_n \cdot \nabla \nu_n &= -\frac{1}{\eta} \nabla p - \frac{1 - x}{x} s \nabla T - \frac{1 - x}{2} \nabla w^2 \\
-(1 - x) w \times (\text{curl} \nu_s) - \frac{1 - x}{\eta} F' - \frac{w \cdot \nabla}{\eta}, \\
\frac{\partial \nu_s}{\partial t} + \nu_s \cdot \nabla \nu_s &= -\frac{1}{\eta} \nabla p + s \nabla T + \frac{x}{2} \nabla w^2 + \\
&+ x w \times (\text{curl} \nu_s) + \frac{F'}{\eta},
\end{aligned}
\]
(where \( \Gamma = \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) \)).

If we accept the equations as given by the variational principle (equations (152)), then \( F' \equiv 0 \), and the equations are the same as Landau's, except for the terms in \( \mathbf{w} \times \text{curl} \, \mathbf{v}_s \). The equation for \( \mathbf{v}_s \) may be put into the form (still assuming \( F' \equiv 0 \))

\[
\frac{\partial \mathbf{v}_s}{\partial t} = -\nabla \left\{ \mathbf{e} + \frac{\rho}{\mathbf{e}} - c^2 \mathbf{T} + \frac{1}{2} \mathbf{v}_s^2 - x(1-\frac{1}{2}x) \mathbf{w}^2 \right\} + \mathbf{w} \times \text{curl} \, \mathbf{v}_s,
\]

and this clearly shows that the equations permit the class of solutions with \( \text{curl} \, \mathbf{v}_s = 0 \); for this class of solutions, the equations are identical with Landau's equations. However, the equations also allow more general solutions for which \( \text{curl} \, \mathbf{v}_s \neq 0 \).

As we saw in the preceding section, any choice of \( \mathbf{F}' \) such that \( \mathbf{c} \cdot \mathbf{F}' = 0 \) gave a set of hydrodynamic equations ((163)) which were consistent with the conservation laws and the requirements of Galilean invariance. It is of particular interest to note that the choice

\[
\mathbf{F}' = \mathbf{e} \mathbf{z} \times \text{curl} \, (\mathbf{v} + \mathbf{z} \mathbf{c}),
\]

or

\[
\mathbf{F}' = -\mathbf{c}_n \mathbf{w} \times \text{curl} \, \mathbf{v}_s,
\]

gives the equations

\[
\frac{\partial \mathbf{v}_n}{\partial t} + \mathbf{v}_n \cdot \nabla \mathbf{v}_n = -\frac{1}{\mathbf{c}} \nabla \rho - \frac{1-x}{x} s \nabla T - \frac{1-x}{2} \nabla \mathbf{w}^2 \]

\[
-\frac{\mathbf{w}}{\mathbf{c}_n} \Gamma,
\]

\[(167)\]
\[ \frac{\partial v_x}{\partial t} + v_x \cdot \nabla v_x = -\frac{1}{\rho} \nabla p + \frac{1}{2} \alpha \nabla T + \frac{1}{2} \alpha \nabla w^2, \]

which are the Landau equations (again, the equations (167) permit solutions with \( \text{curl } v_x = 0 \), but also allow more general solutions).

Although we have obtained the two-fluid equations from the one-fluid equations and the relations (164), we have yet to compare the thermodynamic descriptions. According to the one-fluid theory, the total energy per unit volume is given by

\[ E = \frac{1}{2} \rho \overline{v}^2 + \rho e, \]

where \( de = \frac{p}{\rho^2} dp + T dS - c \cdot dz \). In the two-fluid model, the total energy is

\[ E = \frac{1}{2} \rho_0 \overline{v}^2 + \frac{1}{2} \rho_1 \overline{v}_1^2 + \rho \overline{e}, \]

which defines the specific internal energy \( \overline{e} \) of the two-fluid model. By equating the two expressions for the energy and making use of (164), we find that

\[ \overline{e} = e - \frac{1}{2} \chi (1 - \chi) w^2 = e + \frac{1}{2} \chi c^2, \quad (169) \]

or

\[ \overline{e} = e_p \quad (169) \]

Thus

\[ de = d(e + \frac{1}{2} \chi c^2) \]

\[ = \frac{p}{\rho^2} dp + T dS - \frac{1}{2} c^2 dz \]

\[ = \frac{p}{\rho^2} dp + T dS + \frac{1}{2} w^2 dx, \quad (170) \]

so that the thermodynamic description obtained by transformation from the one-fluid model is the same as that given
originally for the two-fluid model (equation (14) or equation (64)). We may also note now that the extensive variable \( e z c \) in the one-fluid model may be written as

\[-e z c = e \cdot w\]

and thus in the two-fluid model may be interpreted as the (negative of) the momentum per unit volume of the fluid in the superfluid rest frame.

From the above, it is seen that the equations of the one-fluid model are in substantial agreement with Landau's equations for the two-fluid model. The equations (165) (with \( E' = 0 \)) are those given by Lin [24]. These equations differ from Landau's equations by the extra terms in \( w \times \text{curl} \, v_s \), but it would be difficult to decide from experiments whether the perfect fluid equations should contain such terms since it is usually true that dissipative processes are important in flows for which \( \text{curl} \, v_s \neq 0 \). Also, because of the difficulty with the variational principle, it is perhaps slightly preferable to retain the more general equations (165) with \( E' \neq 0 \) (which include the Landau equations (167) as a special case). Perhaps it is well to emphasize again that Landau's method for deriving the two-fluid equations yields a unique result only when the restrictive equation \( \text{curl} \, v_s = 0 \) is imposed. As we saw in section II-A-1, Landau's method fails to yield a unique set of equations if the restrictive equation \( \text{curl} \, v_s = 0 \) is dropped. Thus if we do not insist that \( \text{curl} \, v_s \) must vanish, then the difficulties in obtaining a definite set of
hydrodynamic equations are present in Landau's theory to the same extent as in Lin's one-fluid theory.
C. Discussion

In this section, we give a brief discussion and comparison of the two-fluid model and the one-fluid model, and the various methods used to obtain the hydrodynamic equations for reversibla flows of helium II. We consider first the two-fluid model.

The derivation of the Landau equations for the two-fluid model from the conservation laws (as discussed in II-A-1) has the advantage that it is based on general principles of universal validity (i.e., conservation laws and Galilean invariance). However, the possibility of unambiguously obtaining the Landau equations in this manner depends in an essential way on the additional special requirement that \( \text{curl} \mathbf{v}_2 \) must vanish. If this requirement is not imposed, then the conservation laws and the Galilean relativity principle are not sufficient to determine the hydrodynamic equations uniquely, and some additional principle is needed. In his 1941 paper [20], Landau gave such an additional principle—namely, that there be no momentum exchange between the two components (other than that due to \( \mathbf{p}_n \rightarrow \mathbf{p}_s \) transitions). In a sense, this principle is really the foundation of the two-fluid model because it is the one principle which distinguishes the helium two-fluid system from a mixture of two ordinary substances in which a chemical reaction (or dissociation) is taking place. In order to formulate this principle precisely and use it in the derivation of the hydrodynamic equations, however, it is
clear that a necessary prerequisite is a complete thermodynamic and hydrodynamic description for each of the components of the two-fluid system. The derivation of the hydrodynamic equations given in II-A-2 is based on these ideas, and it was shown there that one may deduce the Landau equations without the restriction \( \text{curl} \, \mathbf{v}_s = 0 \) in an unambiguous manner. Finally, in section II-A-3, a discussion of Zilsel's variational principle for the two-fluid model was given. Although this method yields the Landau equations, it entails certain restrictions on the quantities \( \text{curl} \, \mathbf{v}_a \) and \( \text{curl} \, \mathbf{v}_s \). It is probable that these restrictive conditions do not have physical significance, and that they arise from defects in the variational principle connected with the fact that one cannot introduce a Lagrangian description of the two-fluid system.

For Lin's one-fluid model, two derivations of the hydrodynamic equations were discussed. The first of these, the variational principle discussed in II-B-2, leads to a definite set of hydrodynamic equations, but also implies a restriction on the quantity \( \text{curl} \, \mathbf{a} \). It is possible that this difficulty stems from the fact that the Hamilton's principle of mechanics was used to obtain the equations of motion, whereas we know that even for reversible flows of helium II energy transfer processes of an essentially thermal nature may take place. Lin\(^*\) has suggested that a more thorough study of the thermodynamics of the one-fluid model may reveal a way in which the variational principle should be modified to take these thermal

\[\text{private communication}\]
processes into account. The other derivation of the equations for the one-fluid model (based on the conservation laws and the Galilean relativity principle, section II-B-3) did not yield a unique set of equations (this, of course, is to be compared with an exactly similar result for the two-fluid model when the restriction \( \text{curl} \mathbf{V}_S = 0 \) is not imposed). The class of equations consistent with the conservation laws includes the Landau equations and also the equations originally proposed by Lin \([25,24]\).

As discussed earlier (II-B-4), the equations of the one-fluid model are equivalent in all essential respects with the Landau equations for the two-fluid model. Although the macroscopic equations are essentially the same for the two models, the underlying physical ideas are very different. The origins of the two-fluid model are the various microscopic theories of liquid helium II, while the one-fluid model represents a generalization of ordinary hydrodynamics suggested by macroscopic observations. Since the relationship between the various microscopic theories and the actual molecular structure of helium II is not clear at present, it is reassuring that one may obtain hydrodynamic equations equivalent to those of the two-fluid model from the one-fluid model which has its foundations directly in macroscopic observations.