CHAPTER IV - DISSIPATIVE PROCESSES - II

A. Introduction

In this chapter, we will consider some theories of liquid helium II in which the quantity \( \text{curl} \mathbf{v}_s \) plays a special role. We will be concerned primarily with the continuum theory proposed by Bekarevich and Khalatnikov [3] in which the quantity \( |\text{curl} \mathbf{v}_s| \) is a thermodynamic variable. The derivation of the hydrodynamic equations proposed by Hall and Vinen [12,13,14,38] for the motion of helium II with quantized vortex lines will not be discussed in detail here. However in section IV-C we will discuss briefly an alternative continuum approach to the hydrodynamics of helium II with quantized vortex lines. It is perhaps helpful to recall here the essential features of these theories before beginning a detailed discussion.

In his 1941 paper [20], Landau advanced the idea that the flow of the superfluid component must be potential flow—that is, that \( \text{curl} \mathbf{v}_s = 0 \). The experiments of Andronikašvili [1] with an oscillating disc pile seemed to provide a striking experimental verification of this hypothesis. Later experiments, however, indicated that the superfluid component could rotate in some manner (e.g., the free surface experiments of Osbourne [31]). Onsager [30] and later, independently, Feynman [7] suggested that, although the superfluid component cannot rotate in bulk, it can support line singularities...
analagous to vortex lines in ordinary hydrodynamics. Feynman gave some qualitative quantum-mechanical arguments to support the view that the strength of the vortexes is quantized according to the formula

$$\oint \mathbf{v} \cdot d\mathbf{s} = 2\pi n \frac{\hbar}{m}.$$  

Hall and Vinen [12,13,14,38] then developed a hydrodynamic theory for helium II on the basis of the Onsager-Feynman theory of quantized vortex lines. They were able to obtain a definite set of hydrodynamic equations (a comprehensive review of their work--theoretical and experimental--is given in [14,38]). However, the derivation of Hall and Vinen is dependent in an essential manner on some special additional assumptions about the nature of the vortex motion in helium II (for example, they assume that the force on a vortex line and the velocity of translation of the line relative to the mean superfluid velocity are related via the classical Magnus effect formula). In an attempt to obtain a hydrodynamic theory of vortex motion in helium II which is independent of specific assumptions about the nature of the vortex motion, Bekarevich and Khalatnikov [3] have developed a hydrodynamic theory based only on general continuum principles (conservation laws, increase of entropy, etc.) and the single additional assumption that the thermodynamic internal energy of the helium II depends on $|\text{curl} \mathbf{v}|$, as well as the usual thermodynamic variables. It is noteworthy that their hydrodynamic equations agree with those presented earlier by Hall [14].
Since the form of the hydrodynamic equations for helium II is still a controversial matter, it is desirable that any new phenomenological theory, such as the one proposed by Bekarevich and Khalatnikov, include (i) a careful mathematical development of the hydrodynamic equations based on the general assumptions of the theory and (ii) a discussion of the physical basis of the phenomenological theory. In the presentation of Bekarevich and Khalatnikov [3], however, (i) the mathematical development of the theory seems obscure in places, and (ii) almost no discussion is given of the physical basis of their theory. (More specifically, there are a number of questions connected with their phenomenological theory which deserve discussion; for example: (i) on the basis of the Onsager-Feynman theory of quantized vortex lines, should we expect to be able to develop a continuum theory for describing the motion of helium II, and, if so, is the inclusion of the superfluid vorticity in the thermodynamic variables a plausible starting point for the development of such a theory? (ii) are there other plausible pictures of the microscopic structure of helium II which would also lead to a continuum theory of the form proposed by Bekarevich and Khalatnikov [3]?)

In section IV-B-1, we give a detailed presentation, together with some criticisms, of the derivation of the hydrodynamic equations as presented by Bekarevich and Khalatnikov [3]. In section IV-B-2, we give an alternative mathematical development of their theory which seems in some respects to be more satisfactory. In IV-B-3, we give some discussion of
the physical basis of their theory. Finally in IV-C, we give a brief outline of an alternative approach to the problem of building a hydrodynamic theory of helium II on the basis of the Onsager-Feynman vortex line theory.
B. Theory of Bekarevich and Khalatnikov

1. Bekarevich's and Khalatnikov's derivation of the hydrodynamic equations

According to Bekarevich and Khalatnikov, the fundamental distinction between motions for which \( \nabla \times \vec{v} = 0 \) and those for which \( \nabla \times \vec{v} \neq 0 \) is that, in the latter case, the thermodynamic internal energy of the helium II depends on \(|\nabla \times \vec{v}|\). They have used this idea, together with the usual conservation laws and invariance principles, to obtain a set of hydrodynamic equations for helium II. In this section we present a review of their derivation along with detailed discussions of some points in their work which seem obscure.

The method of derivation used by Bekarevich and Khalatnikov is essentially the same as the general method discussed in Chapter III. Briefly, it is as follows: one writes down the equations expressing the conservation of mass, energy and momentum (where the energy and momentum fluxes are to be determined), and also an equation for the superfluid velocity \( \vec{v}_s \); these equations must yield an equation for the entropy which satisfies the law of increase of entropy, and this requirement leads to a single equation relating the energy flux, momentum flux and the dissipation function to known quantities; from this single equation, one determines the energy flux, momentum flux and dissipation function separately. It is clear that, in order to carry out this procedure (especially the last step), one must have some knowledge of what quantities the unknown fluxes may depend on. In Chapter III, this procedure
was carried out systematically by starting out with the perfect fluid equations, and then assuming that the increments to the various fluxes were linear functions of the independent gradients. In the present case, although the procedure cannot be carried out in quite such a straightforward manner, we would still expect to first develop a perfect fluid theory and then a theory of dissipative processes (this will be carried out in IV-B-2). Bekarevich and Khalatnikov, however, have (apparently) identified rotational flow ($\text{curl } \mathbf{v}_s \neq 0$) with dissipative flow. In their theory, they start from the Landau equations for reversible processes, rather than developing a theory of reversible flows for the case in which the internal energy depends on $|\text{curl } \mathbf{v}_s|$; the Landau equations, however, refer to a different physical system (one in which the internal energy does not depend on the vorticity) so that their procedure does not seem entirely satisfactory.

We now proceed to the details of their derivation. (We have changed the notation somewhat to conform with that used in the present work.) According to the Galilean transformation formula, the total energy per unit volume $E$ may be expressed in terms of the total energy per unit volume as measured in the superfluid rest frame, $E_0$, as

$$E = \frac{1}{2} \rho \mathbf{v}_s^2 + \mathbf{v}_s \cdot \mathbf{j}_s + E_0$$

(273)

where $\rho$ is the total density, $\mathbf{v}_s$ the superfluid velocity and $\mathbf{j}_s$ is the momentum per unit volume in the superfluid rest frame. The total momentum per unit volume $\mathbf{j}$ is related to $\mathbf{j}_s$.
by
\[ j = j_o + \xi \psi. \]  

(274)

In the usual version of the two-fluid model, the differential of the energy \( E_o \) is given by
\[ dE_o = \Phi d\epsilon + T d(\epsilon s) + \mathbf{w} \cdot d\mathbf{j}_o, \]
(cf. equation (9)) where
\[ \epsilon \Phi = E_o - \mathbf{w} \cdot \mathbf{j}_o - T \epsilon s + \pi \]  
(cf. equation (10)). In the present case, there will be an additional term expressing the dependence of the energy on the superfluid vorticity \( \omega \equiv \text{curl} \ \psi \). Thus
\[ dE_o = \Phi d\epsilon + T d(\epsilon s) + \mathbf{w} \cdot d\mathbf{j}_o + \lambda \cdot d\omega. \]  

(275)

Bekarevich and Khalatnikov make the plausible assumption that the differential coefficient \( \lambda \) depends only on the direction of the vector \( \overline{\omega} \) (conceivably, it could depend on the direction of the the vector \( \mathbf{w} \) as well). Then, for
\[ \lambda = \lambda \mathbf{v}, \quad \mathbf{v} = \frac{\omega}{||\omega||}, \]  

(276)

we have
\[ dE_o = \Phi d\epsilon + T d(\epsilon s) + \mathbf{w} \cdot d\mathbf{j}_o + \lambda d\omega. \]  

(277)

(According to the Onsager-Feynman theory of quantized vortex lines, the coefficient \( \lambda \) is given approximately by \( \lambda = \frac{\hbar R}{m} \ln \frac{R}{a} \), where \( R/a \) is the ratio of the distance between vortices to the effective core radius of the vortex. Thus \( \lambda \) is expected to be virtually independent of the vorticity \( \omega \).) The relation between the pressure \( \pi \) and the potential \( \Phi \) is given by Bekarevich and Khalatnikov (without discussion) as
\[ p = -E_0 + T \phi \mathcal{E} + \phi \mathcal{F} + \mathbf{w} \cdot \mathbf{j}_0. \]  

(278)

This is the same as equation (10) relating \( p \) and \( \phi \) for the ordinary two-fluid model. If the vorticity \( \omega \) is taken as an extensive quantity per unit volume* (as the formula (275) would imply), one would expect that \( p \) and \( \phi \) are related by 

\[ p = -E_0 + T \phi \mathcal{E} + \phi \mathcal{F} + \mathbf{w} \cdot \mathbf{j}_0 + \lambda \omega, \]  

rather than by (278). In fact, Bekarevich and Khalatnikov find it convenient at a later point in their derivation to introduce a "renormalized" pressure, \( p_0 = p + \lambda \omega \) (with \( p \) given by (278)). In section IV-B-2, we will give a more detailed discussion of the thermodynamics of the present system; for now, we accept (277) and (278), and continue with the derivation of Bekarevich and Khalatnikov.

The equations expressing the conservation of mass, energy and momentum are 

\[ \frac{\partial \rho}{\partial t} + \text{div} \mathbf{j} = 0, \]  

(279)

\[ \frac{\partial E}{\partial t} + \text{div} (Q_0 + q) = 0, \]  

(280)

and 

\[ \frac{\partial j}{\partial t} + \text{div} (\Pi^0 + \Pi) = 0, \]  

(281)

where \( Q_0 \) and \( \Pi^0 \) are the energy flux vector and momentum flux tensor of the Landau theory for irreversible processes, namely 

\[ Q_0 = (\phi + \frac{1}{2} \mathbf{v}_s^2) \mathbf{j} + \mathcal{E} \mathcal{s} T \mathbf{v}_n + \mathcal{V}_n (\mathbf{v}_n \cdot \mathbf{j}_0). \]  

(282)

* According to the Onsager-Feynman theory, the total length of vortex line in a volume \( V \) is given by \( \int dt \omega / k \), where \( k = h / m \), is the circulation around each line; thus \( \omega \) is proportional to the total length of vortex line per unit volume.
\[ \pi^i_j = \rho \nu_{si} \nu_{sj} + \nu_{si} j_{oj} + \nu_{sj} j_{oi} + \rho \delta_{ij}, \quad (283) \]

or

\[ \pi^i_j = \rho \nu_{ni} \nu_{nj} + \rho \nu_{si} \nu_{sj} + \rho \delta_{ij} \quad (284) \]

according to Bekarevich and Khalatnikov, the increments \( \pi \) and \( \nu \) represent the effects of the dissipative processes (whereby they apparently mean the effects of the dissipative processes and the effect on the structure of the equations of including the vorticity in the thermodynamic variables). There will also be an equation for the superfluid velocity \( \nu_s \); this equation may be written as:

\[ \frac{\partial \nu_s}{\partial t} + \nu_s \cdot \nabla \nu_s + \nabla \Phi = f, \quad (285) \]

where the quantity \( f \) is to be determined. Finally, there will be an equation for the entropy \( \rho s \) of the form

\[ \frac{\partial}{\partial t} (\rho s) + \text{div} \rho s \nu_n = \frac{R}{T}, \quad (286) \]

where the dissipation function \( R \) is to be positive definite. (One should really include a contribution to the entropy flux from dissipative processes; as in the theories discussed in Chapter III, such a term is associated with the process of thermal conduction which, as Bekarevich and Khalatnikov point out, is easily included in their theory. They also point out that their theory is easily modified to include dissipative gradient terms in the superfluid equation such as those...
discussed in section III-B-2 (cf. equation (175) and (189).)
The entropy equation (286) is not independent, but is a consequence of the hydrodynamic equations. By eliminating the time derivatives in the usual manner, we may obtain the following equation relating the quantities \( f \), \( \Pi \), \( \varrho \) and \( R \):

\[
\text{div}\left\{-\varrho + \Pi \cdot \nabla + \lambda \nabla \times [\mathbf{f} + \omega \times \mathbf{w}]\right\} = R + \left( \Pi_{ik} - \lambda \omega \delta_{ik} + \lambda \omega_i \omega_k \right) \frac{\partial \nu_{ni}}{\partial x_k} + (f + \omega \times \mathbf{w}) \cdot (\text{curl} \lambda \nabla - \varphi_\varepsilon \mathbf{w}).
\]  

(287)

At this point, the authors conclude (without giving any justification) that

\[
\varrho = \Pi \cdot \nabla + \lambda \nabla \times (f + \omega \times w),
\]  

(288)

and

\[
R = -\left[ \Pi_{ik} - \lambda \omega_i \omega_k - \lambda \omega \delta_{ik} \right] \frac{\partial \nu_{ni}}{\partial x_k} - (f + \omega \times \mathbf{w}) \cdot (\text{curl} \lambda \nabla - \varphi_\varepsilon \mathbf{w}).
\]

In order to obtain the results (228) and (289) from (287), one would have to give careful arguments concerning the dependence of the (as yet unknown) quantities \( f \), \( \Pi \) and \( \varrho \) on the macroscopic variables and their gradients. It is easy enough to find special choices of \( f \), \( \Pi \) and \( \varrho \) which satisfy (287) without satisfying (288) and (289). Rather than trying to find principles which would allow one to obtain (288) and (289) from (287), we prefer to leave this point for the more systematic derivation of the equations given in the next section; thus we accept (288) and (289), and continue with the derivation of Bekarevich and Khalatnikov.

The dissipation function \( R \) is to be positive definite.
Bekarevich and Khalatnikov, however, assume that each of the
two terms in $\mathbb{R}$ are separately positive definite; again, they
give no justification and counter-examples may be easily
found. If we make this assumption, then we must have

$$-(f + \omega \times w) \cdot (\text{curl} \lambda \nu - \varphi_5 w) \geq 0,$$

(290)

and

$$-(\pi_{ik} + \lambda \omega \delta_{ik} + \frac{\lambda \omega w}{\omega}) \frac{\partial v_{ni}}{\partial x_k} \geq 0.$$  

(291)

From (290), Bekarevich and Khalatnikov conclude (without jus-
tification) that the most general form for $f$ is

$$f = -\omega \times w + \alpha \omega \times (\text{curl} \lambda \nu - \varphi_5 w) + \beta \nu \times [\omega \times (\text{curl} \lambda \nu - \varphi_5 w)]$$

$$-\gamma \nu [\omega \cdot (\text{curl} \lambda \nu - \varphi_5 w)],$$

(292)

with $\beta, \gamma \geq 0$. Here, we may make a more definite criticism
of their reasoning. First of all, it is clear that we can say
almost nothing about the most general form for $f$ unless we
know what vectors $f$ may depend on. However, it would seem
reasonable to suppose that $f$ may depend on $w$, $\omega$ and $\text{curl} \lambda \nu$
(as well as any Galilean invariant scalars). Then it is a
straightforward matter to show that the most general $f$ satis-
fying (290) and depending on $w$, $\omega$ and $\text{curl} \lambda \nu$ is given by
(where $p = \text{curl} \lambda \nu - \varphi_5 w$)

$$f = -\omega \times w - c_0 p + p \times \left\{ p \times \left[ c_1 w + c_2 \omega \times w + c_3 \omega \times p + c_4 \omega \times (\omega \times w) + c_5 \omega \times (\omega \times p) \right] \right\}$$

(293)
where \( c_0 > 0 \), \( c_i (i = 1, \ldots, 5) \) are arbitrary, and where the \( c' \)s in general will depend on all of the scalar invariants of the vectors \( \omega \), \( \mathbf{w} \) and \( \text{curl} \lambda \mathbf{v} \). The expression (293) for \( \mathbf{f} \) is clearly much more general than the expression (292). If we assume further that the sum \( \mathbf{f} + \omega \times \mathbf{w} \) does not depend on the vector \( \mathbf{w} \), then the most general \( \mathbf{f} \) such that \( \mathbf{f} + \omega \times \mathbf{w} \) depends only on \( \mathbf{p} \) and \( \omega \) and such that (290) is satisfied is given by

\[
\mathbf{f} + \omega \times \mathbf{w} = -c_0 \mathbf{p} + \mathbf{p} \times \left[ \mathbf{p} \times \left\{ c_3 \omega \times \mathbf{p} + c_5 \omega \times (\omega \times \mathbf{p}) \right\} \right],
\]

(294)

where \( c_0 > 0 \), \( c_3 \) and \( c_5 \) are functions of the scalar invariants of \( \mathbf{p} \) and \( \omega \) (and of the thermodynamic variables). If now we choose \( c_0 \), \( c_3 \) and \( c_5 \) as

\[
c_0 = \frac{\alpha}{\omega p^2} (\omega \cdot \mathbf{p})^2 + \frac{\beta}{\omega p^2} (\omega \times \mathbf{p})^2,
\]

\[
c_3 = \frac{\alpha}{\omega p^2}, \quad \text{and} \quad c_5 = \frac{\gamma - \beta}{\omega p^2}
\]

where \( \alpha \), \( \beta \), \( \gamma \) are scalar functions and \( \beta > 0 \), \( \gamma > 0 \), then (294) reduces to

\[
\mathbf{f} = -\omega \times \mathbf{w} + \alpha \omega \times \mathbf{p} + \beta \gamma \times \{ \omega \times \mathbf{p} \} - \gamma \gamma (\omega \cdot \mathbf{p}),
\]

(295)

which is the expression (292) given by Bekarevich and Khalatnikov. Thus (292) is a special case of (294); we may note that (292) is truly a special case since not every vector of the type (294) may be written in the form (292) with \( \beta > 0 \), \( \gamma > 0 \). A more serious criticism is that we have had to assume that the combination \( \mathbf{f} + \omega \times \mathbf{w} \) does not depend on \( \mathbf{w} \);
this would seem to be very unnatural, since \( \mathbf{f} + \mathbf{\omega} \times \mathbf{w} \) is the quantity of physical significance. Although Bekarevich and Khalatnikov may have had reasons for choosing the special form (292), it is clear from the above discussion that some justification for this choice is called for.

Now consider the tensor terms; if we introduce \( \tau_{ik} \) by

\[
\Pi_{ik} = \tilde{\Pi}_{ik} - \tau_{ik},
\]

where

\[
\tilde{\Pi}_{ik} = \lambda \omega \delta_{ik} - \frac{\lambda \omega \cdot \omega}{\omega},
\]

then the condition (291) is

\[
\tau_{ik} \frac{\partial \nu_{ki}}{\partial x_k} \geq 0.
\]

Thus \( \tau_{ik} \) must have the form

\[
\tau_{ik} = \mu_{iklm} \frac{\partial \nu_{ni}}{\partial x_m}.
\]

In principle, the viscosity tensor \( \mu_{iklm} \) could depend on \( \omega \) (as pointed out by Bekarevich and Khalatnikov) and on the relative velocity \( \mathbf{w} \) as well. As a first approximation, however, we may take \( \mu_{iklm} \) to be isotropic so that the stress tensor \( \tau_{ik} \) is given by

\[
\tau_{ik} = 2 \mu \left\{ e_{ij}^{(n)} - \frac{1}{3} \delta_{ij} e_{kk}^{(n)} \right\} + \lambda \mathbf{e}_{kk}^{(n)} \mathbf{S}_{ij}.
\]

(296)

This completes the derivation of the hydrodynamic equations.
as given by Bekarevich and Khalatnikov. The final equations for the normal fluid component, the superfluid component and the momentum are (where $\beta_{s}' = 1 + \alpha_{s}$)

\[
\frac{\partial \nu_n + \nu_n \cdot \nabla \nu_n}{\partial t} = -\nabla P_0 + (1-x) \nabla \nabla \nu - \nabla x \frac{\omega}{\xi} \nabla \lambda \nu + \frac{\gamma}{\xi} \frac{\lambda}{p_s} \nabla \nu + \nabla \frac{\xi}{p_s} \frac{\gamma}{\xi} \nabla x \left\{ \nabla \left( \nabla \lambda \nu \right) \right\} + \frac{\beta_{s}^2}{p_s} \frac{\gamma}{\xi} \nabla x \left\{ \nabla \left( \nabla \lambda \nu \right) \right\} \frac{\xi}{p_s} + \frac{\gamma}{\xi} \frac{\lambda}{p_s} \nabla \nu \times \left\{ \nabla \left( \nabla \lambda \nu \right) \right\} \frac{1}{p_s}
\]

\[(298)\]

\[
\frac{\partial \nu_s + \nu_s \cdot \nabla \nu_s}{\partial t} = -\nabla P_0 + (1-x) \frac{\lambda}{\xi} \nabla \nu - \frac{\gamma}{\xi} \frac{\lambda}{p_s} \nabla \nu \times \left\{ \nabla \left( \nabla \lambda \nu \right) \right\} + \frac{\beta_{s}^2}{p_s} \frac{\gamma}{\xi} \nabla x \left\{ \nabla \left( \nabla \lambda \nu \right) \right\} \frac{1}{p_s} + \frac{\gamma}{\xi} \frac{\lambda}{p_s} \nabla \nu \times \left\{ \nabla \left( \nabla \lambda \nu \right) \right\} \frac{1}{p_s}
\]

\[(299)\]

and

\[
\frac{\partial j_i + \partial}{\partial x_j} \left\{ p_n \nu_i \nu_j + p_s \nu_i \nu_j + p_s \delta_{ij} - \frac{\lambda \omega_i \omega_j}{\omega} \tau_{ij} \right\} = 0 \quad (300)
\]

where

\[p_0 = p + \lambda \omega.\]

There are several additional points which we would like to note here. First of all, we see from the expression (288) for the increment to the energy flux that the additional stresses due to the vortex motion (namely the terms \(\pi_{ij} = \lambda \omega \delta_{ij} - \frac{\lambda \omega_i \omega_j}{\omega} \) in \(\pi_{ij}\)) are a part of the normal fluid stress.
tensor. This is somewhat surprising, since it is the vorticity of the superfluid component which plays a special role here (we shall see in the next section that this part of the stress tensor actually should be associated with the superfluid component). Another hint that something is amiss is the fact that Bekarevich and Khalatnikov have given a physical interpretation of the term \( \mathbf{\Pi} \cdot \mathbf{v} + \lambda \nabla \times (\mathbf{f} + \omega \times \mathbf{w}) \) in the energy flux in terms of the transport of energy by the vortex motion. However, the term \( \mathbf{\Pi} \cdot \mathbf{v} \) already has an interpretation as the rate of working of the stress tensor \( \mathbf{\Pi} \); thus what is needed is a physical interpretation of the term \( \lambda \nabla \times (\mathbf{f} + \omega \times \mathbf{w}) \) alone. This point will be discussed in more detail in the next section.

In situations of practical interest, there are a number of approximations which can be made. First of all, for flows at moderate speeds we may take the fluid to be incompressible; the quantities \( \rho_0, \rho, \sigma \) are then all approximately constant, and \( \mathbf{p}, \mathbf{T} \) become hydrodynamic variables. Also, on the basis of the microscopic theory (the Onsager-Feynman theory) the coefficient \( \lambda \) should be approximately constant. According to Bekarevich and Khalatnikov, the term in the mutual friction having the coefficient \( \gamma \) represents the effect of deviations of the direction of individual vortex filaments from the direction of the mean curl of the velocity \( \mathbf{w} \); they state that the effect is very small so that \( \gamma \) is negligible in comparison with \( \beta, \beta' \). A further simplification is that we may neglect the energy dissipation for flows
of moderate velocities. When these approximations are made, the equations take the form

\[
\frac{\partial \mathbf{v}_n}{\partial t} + \mathbf{v}_n \cdot \nabla \mathbf{v}_n = -\nabla p_n - \frac{(1-x)}{x} \mathbf{s} \nabla T + \frac{1}{2} \frac{\rho_n}{\rho_p} B' \left\{ \mathbf{w} \times \left( \mathbf{w} - \frac{\text{curl} \lambda \mathbf{v}}{\rho_p} \right) \right\} \\
+ \mu_n \nabla^2 \mathbf{v}_n + \frac{1}{2} \frac{\rho_n}{\rho_p} B \nabla \times \left\{ \mathbf{w} \times \left( \mathbf{w} - \frac{\text{curl} \lambda \mathbf{v}}{\rho_p} \right) \right\}, \tag{301}
\]

\[
\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s = -\nabla p_s + \mathbf{s} \nabla T + \mathbf{v} \nabla \frac{1}{2} w^2 - \frac{1}{2} \mathbf{w} \times \text{curl} \lambda \mathbf{v} \\
- \frac{1}{2} \frac{\rho_n}{\rho_p} B' \left\{ \mathbf{w} \times \left( \mathbf{w} - \frac{\text{curl} \lambda \mathbf{v}}{\rho_p} \right) \right\} - \frac{1}{2} \frac{\rho_n}{\rho_p} B \left\{ \mathbf{w} \times \left( \mathbf{w} - \frac{\text{curl} \lambda \mathbf{v}}{\rho_p} \right) \right\}, \tag{302}
\]

and

\[
\text{div } \mathbf{v}_n = 0, \quad \text{div } \mathbf{v}_s = 0, \tag{303}
\]

where

\[
\rho_n = \rho + \lambda \omega, \quad \beta = \frac{1}{2} \frac{B}{\rho_p} \rho_n, \quad \beta' = \frac{1}{2} \frac{B'}{\rho_p} \rho_n.
\]

The equation for the total momentum is

\[
\frac{\partial}{\partial t} \left\{ \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s \right\} + \frac{\partial}{\partial x_i} \left\{ \rho_n \mathbf{v}_n \mathbf{v}_n + \rho_s \mathbf{v}_s \mathbf{v}_s \mathbf{v}_j + \rho_n \mathbf{S}_{ij} - \frac{\lambda \omega_i \omega_j}{\omega} \\
- 2 \mu_n \mathbf{e}_{ij}^{(m)} \right\} = 0. \tag{304}
\]

The equations (301) and (303) are in agreement with the equations obtained by Hall [14] (there is apparently a computational error in equation (51) of reference [14]; the results therefore are in agreement with the equations given by Bekarevich and Khalatnikov provided this is taken into account). The equation for the superfluid vorticity is also of interest;
when the coefficient $\gamma$ is taken to be zero, this equation may be written as

$$\frac{\partial \omega}{\partial t} = \text{curl} \left\{ f + v_s \times \omega \right\} = \text{curl} \left\{ v_L \times \omega \right\}, \quad (305)$$

where $v_L = v_s + \frac{1}{\rho_s} \text{curl} \lambda \nabla + \frac{1}{2} B \frac{\rho_n}{\rho} \left\{ w - \frac{1}{\rho_s} \text{curl} \lambda \nabla \right\} + \frac{1}{2} B \frac{\rho_n}{\rho} \nabla \times \left( w - \frac{1}{\rho_s} \text{curl} \lambda \nabla \right), \quad (306)$

Equation (305) has the form of a transport equation with the velocity of transport $v_L$ being given by (306).

Finally, Bekarevich and Khalatnikov have given a discussion of the boundary conditions which we review briefly here. The condition on $v_n$ at a solid surface is taken to be the usual one; thus at a wall moving with velocity $U_{\text{wall}}$, we have

$$v_n = U_{\text{wall}}. \quad (307)$$

(The present discussion is limited to the case in which there is no heat current through the wall, so that $v_n$ and $v_s$ satisfy $(v_n - U_{\text{wall}}) \cdot n = 0$, $(v_s - U_{\text{wall}}) \cdot n = 0$ and so that the thermal conduction term in the energy flux is unimportant). Since the equation for $v_s$ contains second space derivatives of $v_s$, a further boundary condition is needed. Bekarevich and Khalatnikov have shown how to obtain the general form of this boundary condition from the requirement that the rate of energy dissipation be positive definite. Let $n$ be the unit
normal pointing from the fluid into the solid. Then the force
(per unit area) exerted on the solid by the fluid is

\[ F = (τ_{ij} + ρ\dot{ε}_{ij})n_j. \]

The rate of working of the stresses is then (taking \( u_{\text{wall}} \cdot n = 0 \))

\[ F \cdot u_{\text{wall}} = τ_{ij} u_{\text{wall} i} n_j. \]

The rate at which energy flows (per unit area) from the fluid
to the solid is

\[ Q \cdot n = \tau_{ij} n_i \nabla n_j + \lambda n \cdot \{ \nabla \times (f + \omega \times \nu) \}. \]

Thus the rate of energy dissipation is given by

\[ D = q \cdot n - F \cdot u_{\text{wall}}, \]

or since

\[ \nu n = u_{\text{wall}}, \quad D = \lambda n \cdot \{ \nabla \times (f + \omega \times \nu) \} \]

or

\[ D = \lambda (n \times \nu) \cdot (f + \omega \times \nu). \]

From the condition \( D > 0 \), the authors conclude that

\[ f + \omega \times \nu \bigg|_{\text{wall}} = \mathbf{g} (n \times \omega) + \mathbf{g}' \{ \omega \times (\nu \times n) \}, \]

where (presumably) \( \mathbf{g}, \mathbf{g}' \) are material constants and \( \mathbf{g} \geq 0 \).

We may attempt to supply the reasoning as follows: the quan-
tity \( (f + \omega \times \nu) \bigg|_{\text{wall}} \) must be such as to insure that \( D \) is
positive definite; if we assume that \( (f + \omega \times \nu) \bigg|_{\text{wall}} \) depends
only on the vectors $\mathbf{n}$ and $\mathbf{u}_{\text{wall}}$, then the general form for $\mathbf{f} + \mathbf{u} \times \mathbf{w}$ may be written as

$$\mathbf{f} + \mathbf{u} \times \mathbf{w} \bigg|_{\text{wall}} = \mathbf{s} (\mathbf{n} \times \mathbf{u}) + \mathbf{s}' (\mathbf{u} \times \mathbf{v} \times \mathbf{n}) + \mathbf{s}'' \mathbf{n}, \quad (310)$$

where $\mathbf{s}$, $\mathbf{s}'$ and $\mathbf{s}''$ may depend on the invariant $\mathbf{v} \cdot \mathbf{n}$ as well as on the scalar thermodynamic variables. (There is no reason why the quantity $\mathbf{f} + \mathbf{u} \times \mathbf{w} \bigg|_{\text{wall}}$ should be independent of the vector $\mathbf{w}$, as Bekarevich and Khalatnikov have apparently assumed; if we allow such a dependence, the expression (310) is replaced by a much more general one.) The condition $\mathbf{D} \geq 0$ requires that $\mathbf{s} \geq 0$. We may obtain the result of Bekarevich and Khalatnikov by assuming that $\mathbf{s}'' = 0$ and that $\mathbf{s}$, $\mathbf{s}'$ are independent of $\mathbf{v} \cdot \mathbf{n}$. The boundary conditions (309) may be written in terms of the vortex velocity $\mathbf{v}_l$ (in the case $\mathbf{v} = 0$); thus

$$\left( \mathbf{v}_l - \mathbf{u}_{\text{wall}} \right) \times \mathbf{w} \bigg|_{\text{wall}} = \mathbf{s} (\mathbf{n} \times \mathbf{w}) + \mathbf{s}' (\mathbf{w} \times (\mathbf{v} \times \mathbf{n})), \quad (311)$$

or, introducing

$$(\mathbf{v}_l - \mathbf{u}_{\text{wall}})_{\text{transverse}} = \mathbf{v} \times (\mathbf{v} \times (\mathbf{v} \times \mathbf{n})), \quad (311)$$

we have

$$(\mathbf{v}_l - \mathbf{u}_{\text{wall}})_{\text{transverse}} \bigg|_{\text{wall}} = \mathbf{s} \mathbf{v} \times (\mathbf{n} \times \mathbf{v}) + \mathbf{s}' (\mathbf{n} \times \mathbf{v}), \quad (311)$$

According to Bekarevich and Khalatnikov, the limiting case $\mathbf{s}$, $\mathbf{s}' \rightarrow 0$ corresponds to an absolutely rough surface. (Thus in their theory, the parameters $\mathbf{s}$, $\mathbf{s}'$ will presumably
depend on the condition of the wall surface.)

This completes the review of the derivation given by Bekarevich and Khalatnikov. In the next section, we will give an alternative derivation of the hydrodynamic equations which provides tentative answers to some of the questions raised here.
2. An alternative development of the equations

In this section we give an alternative derivation of the hydrodynamic equations for helium II. The starting point is the same as in the theory of Bekarevich and Khalatnikov—namely, the two-fluid model with the additional feature that the thermodynamic internal energy depends on the superfluid vorticity. As pointed out in the preceding section, Bekarevich and Khalatnikov have identified rotational flow with dissipative flow. We prefer to distinguish clearly between rotational flow and dissipative flow; thus we begin the derivation by first obtaining the equations for reversible processes in the case when the internal energy depends on $|\text{curl} \, \mathbf{v}_s|$.

As a helpful preliminary, we consider first the case of an ordinary fluid in which the internal energy depends on the vorticity. The state of the system is described by the mass per unit volume $\varrho$, the entropy per unit mass, $S$, the fluid velocity $\mathbf{v}$, and the vorticity $\omega = \text{curl} \, \mathbf{v}$. The total energy per unit volume $E$ is given by

$$E = \frac{1}{2} \varrho \dot{\mathbf{v}}^2 + \varrho e,$$

where the internal energy $e$ is a function of $\varrho$, $S$, and $\omega$. The pressure $p$ and temperature $T$ are then given by

$$de = \frac{p}{\varrho^2} \, d\varrho + T \, dS + \Lambda \cdot d\left(\frac{\omega}{\varrho}\right).$$

Thus, as a thermodynamic variable, $\omega$ is taken to be an
extensive quantity per unit volume. Since the energy \( e \) is a scalar, we must have

\[
\lambda = \beta \omega \tag{314}
\]

where \( \omega \) is a scalar function of \( \rho, s \) and \( \omega^2 \). The hydrodynamic equations must include the conservation laws for mass, momentum and energy; thus

\[
\frac{\partial \rho}{\partial t} + \text{div} \, \rho \mathbf{v} = 0 \tag{315}
\]

\[
\frac{\partial E}{\partial t} + \text{div} \, \text{div} \, \mathbf{Q} = 0 \tag{316}
\]

and

\[
\frac{\partial}{\partial t} (\rho \mathbf{v}) + \text{div} (\rho \mathbf{v} \mathbf{v}) = \text{div} \, \mathbf{\sigma} \tag{317}
\]

where \( \mathbf{Q} \) is the energy flux vector and \( \mathbf{\sigma} \) is the stress tensor. Since we are considering only reversible processes, the entropy must be conserved, so we have

\[
\frac{\partial}{\partial t} (\rho s) + \text{div} (\rho s \mathbf{v}) = 0 \tag{318}
\]

In the determination of \( \mathbf{\sigma} \) and \( \mathbf{Q} \), it is convenient to introduce the quantities \( \mathbf{Q}_0 \) and \( \mathbf{T} \) by

\[
\mathbf{Q} = \mathbf{E}_s - \mathbf{\sigma} \cdot \mathbf{v} + \left[ \frac{\omega}{\rho} \times \text{curl} \, \lambda \right] \times \lambda + \mathbf{Q}_0 \tag{319}
\]

and

\[
\mathbf{\sigma} = \beta \omega \omega - \rho \mathbf{T} + \mathbf{T} \tag{320}
\]
The invariance of the equations under Galilean transformations requires that $Q_0$ and $\mathcal{L}$ be Galilean invariant. The equations (315) - (318) are not all independent, and the requirement that they be consistent may be written in the form (after some calculations)

$$\text{div} \, Q_0 + \lambda \cdot \text{curl} \left( \frac{1}{\epsilon} \text{div} \, \mathcal{L} \right) - \mathcal{L} : \text{grad} \, \nu + \lambda \cdot (\nabla T \times \nabla S) \equiv 0. \quad (321)$$

We now consider the special case in which the entropy is uniform in space (this includes the problem of eventual interest, since the superfluid component of helium II has no entropy). Then (321) becomes

$$\text{div} \, Q_0 + \lambda \cdot \text{curl} \left( \frac{1}{\epsilon} \text{div} \, \mathcal{L} \right) - \mathcal{L} : \text{grad} \, \nu \equiv 0. \quad (322)$$

The simplest "solution" of (322) is to take $\mathcal{L} \equiv 0$, $Q_0 \equiv 0$, and we will do this here (it is conceivable that other choices of $Q_0$ and $\mathcal{L}$ will satisfy (322); however, the physical interpretation given below lends weight to the choice made here).

Then the complete set of hydrodynamic equations is

$$\frac{\partial p}{\partial t} + \text{div} \, p \, \nu = 0, \quad (323)$$

$$\frac{\partial}{\partial t} \left( p \, \nu \right) + \text{div} \left( p \, \nu \nu \right) = \text{div} \left( \lambda \, \omega - p \mathcal{L} \right), \quad (323)$$

or

$$p \frac{D \nu}{Dt} = -\nabla p + \omega \cdot \nabla \lambda \quad (324)$$

or -
\[
\frac{\partial \Phi}{\partial t} + \nu \cdot \nabla \Phi = -\nabla \Phi - \frac{\omega}{\rho} \times \text{curl} \lambda,
\]

where
\[
\Phi = \varepsilon + \frac{p}{\rho} - \lambda \cdot \frac{\omega}{\rho}, \quad (325)
\]

and
\[
\frac{\partial \Phi}{\partial t} + \text{div} \left\{ \nabla \left[ -\left( \lambda \cdot \omega - \rho \frac{\mathbb{I}}{2} \right) \nu + \left( \frac{\omega}{\rho} \times \text{curl} \lambda \right) \times \lambda \right] \right\} = 0. \quad (327)
\]

The equation for the vorticity is
\[
\frac{\partial \omega}{\partial t} = \text{curl} \left\{ \nu \times \omega + \frac{1}{\rho} \text{curl} \lambda \times \omega \right\}, \quad (328)
\]
or
\[
\frac{\partial \omega}{\partial t} = \text{curl} \left\{ \nu_\perp \times \omega \right\},
\]

where
\[
\nu_\perp = \nu + \frac{1}{\rho} \text{curl} \lambda. \quad (329)
\]

Thus we see that there is an additional term in the stress tensor, \(\lambda \cdot \omega\), and that the vorticity satisfies a transport equation with a transport velocity \(\nu_\perp\) given by (329). In the energy flux, there is a convection term, \(\nabla \nu\), a term giving the rate of working of the stresses, \(-\frac{\omega}{\rho} \cdot \nu\), and an additional term, \(\left( \frac{\omega}{\rho} \times \text{curl} \lambda \right) \times \lambda\). We may give a physical interpretation of this term as follows: from the vorticity equation, we have
\[
\frac{d}{dt} \int_{\Omega} \omega \, d\tau = \int_{\Gamma} \mathbf{n} \times (\nu_\perp \times \omega),
\]
where \( \mathbf{\nu} \) is a volume fixed in space, bounded by \( \mathbf{s} \); thus the rate at which vorticity crosses \( \mathbf{s} \) per unit area is \( -\mathbf{n} \times (\mathbf{\nu} \times \mathbf{\omega}) \), so the transport of energy from this source is
\[
-\lambda \cdot \mathbf{n} \times (\mathbf{\nu} \times \mathbf{\omega}) = \mathbf{n} \cdot \lambda \times (\mathbf{\nu} \times \mathbf{\omega}).
\]

The flux of energy relative to the moving fluid is then given by
\[
\mathbf{n} \cdot \lambda \times \left[ (\mathbf{\nu}_s - \mathbf{\nu}) \times \mathbf{\omega} \right] = \mathbf{n} \cdot \left\{ \frac{1}{\varepsilon} \text{curl} \lambda \times \mathbf{\omega} \right\} \times \lambda
\]
in agreement with \( \mathbf{n} \cdot \mathbf{Q} \mid_{\text{fluid rest frame}} \). This completes the discussion of the case of an ordinary fluid, and we now consider the application of these results to the two-fluid model.

To derive the perfect fluid equations, we will use a modification of the method discussed in II-A-2, since this method yields a unique set of equations even in the case when \( \text{curl} \; \mathbf{\nu}_s \neq 0 \). Since the method is discussed in detail there, we give only a brief discussion here. As a starting point for the derivation, we assume that each of the two components has a complete thermodynamic description and that the only coupling between them is due to \( \rho_n \leftrightarrow \rho_s \) transitions. Thus the total energies per unit volume of the superfluid and normal fluid components are given by

\[
\varepsilon_n = \frac{1}{2} \rho_n \mathbf{\nu}_n^2 + \rho_n \varepsilon_n,
\]
and
\[
\varepsilon_s = \frac{1}{2} \rho_s \mathbf{\nu}_s^2 + \rho_s \varepsilon_s.
\]
The specific internal energy $e_n$ is a function of $e_n$ and $s_n$, and

$$d e_n = \frac{P_n}{\rho_n} d \rho_n + T n d S_n, \quad (331)$$

which defines the normal fluid pressure $P_n$ and temperature $T_n$. The energy $e_s$ is a function of $\omega = \operatorname{curl} \mathbf{v}_s$ and $\rho_s$ (we are assuming the entropy of the superfluid component to be zero, as usual), and

$$d e_s = \frac{P_s}{\rho_s^2} d \rho_s + \lambda \cdot d \left( \frac{\omega}{\rho_s} \right), \quad (332)$$

which defines superfluid pressure $P_s$; since $e_s$ is a scalar function of $\rho_s$, $\omega$, we must have

$$\lambda = \gamma \lambda \quad (333)$$

where

$$\gamma = \frac{\omega}{|\omega|} \quad (334)$$

and $\lambda$ is a scalar function of $\rho_s$ and $\frac{1}{2} \omega \gamma$.

The only coupling between the two components is (by assumption) that due to $e_n \rightarrow e_s$ transitions, and we assume further that the system is in equilibrium with respect to this process. The equilibrium condition is obtained from an energy minimum principle. If we consider a fixed volume $V$, the total energy is

$$\mathcal{E} = V (e_n + e_s),$$
the total momentum is \( J = V (p_n v_n + p_s v_s) \), the total entropy is \( S = V (e_n s_n) \), and the total mass is \( M = V (p_n + p_s) \). As discussed in Chapter II, the equilibrium is also to be taken at constant \( v_s \); for the same reasons, we also take the equilibrium at constant \( \mathbf{w} \). Thus the equilibrium condition is given by

\[
\frac{d\mathcal{E}}{d\mathbf{v}, J, M, S, v_s, \mathbf{w}} = 0,
\]

(335)

and this yields the condition

\[
\Phi_n - \Phi_s = \frac{1}{2} \mathbf{w}^2,
\]

(336)

where

\[
\Phi_n = e_n + \frac{p_n}{e_n} - T_n s_n,
\]

and

\[
\Phi_s = e_s + \frac{p_s}{e_s} - \frac{\lambda \cdot \mathbf{w}}{e_s}.
\]

(337)

If we introduce the entropy per unit total mass \( s = (p_s s_n) \), and internal energy \( e \) per unit mass \( (p_e = e_n e_n + e_s e_s) \), then

\[
d e = T_n d s + \frac{p_n + p_s}{e^2} d p + (\Phi_n - \Phi_s) d x + \lambda \cdot d \left( \frac{\mathbf{w}}{e} \right).
\]

(338)

If we introduce the total pressure \( P \) by

\[
P = p_n + p_s,
\]

(339)

and the temperature \( T = T_n \), then (338) becomes

\[
d e = T d s + \frac{P}{e^2} d p + \frac{1}{2} \mathbf{w}^2 d x + \lambda \cdot d \left( \frac{\mathbf{w}}{e} \right).
\]

(340)
For later convenience, we note here that the differentials of the partial pressures $p_n$ and $p_s$ are given by

\[ \frac{dp_n}{\rho_n} = \frac{dp}{\rho} + (1-x) d \frac{1}{2} w^2 + (1-x) \frac{\omega}{\rho_s} \cdot dA, \quad (341) \]

and

\[ \frac{dp_s}{\rho_s} = \frac{dp}{\rho} - x d \frac{1}{2} w^2 - s dT + x \frac{\omega}{\rho_s} \cdot dA. \quad (342) \]

We now consider the hydrodynamic equations for reversible processes. First of all, there will be two continuity equations which, however, must take into account the transitions which conserve only the total mass. Thus

\[ \frac{\partial p_n}{\partial t} + \text{div} (p_n v_n) = \Gamma, \quad (343) \]

and

\[ \frac{\partial p_s}{\partial t} + \text{div} (p_s v_s) = -\Gamma, \]

where the volume rate of conversion $\Gamma$ is not to be independently specified, but is determined by the flow conditions and the equilibrium condition (336). Since we are considering only reversible processes, the entropy is conserved and we have

\[ \frac{\partial}{\partial t} (p_n s_n) + \text{div} (p_n s_n v_n) = 0. \quad (344) \]

To obtain the momentum equations, we simply assume that each component satisfies an appropriate perfect fluid equation,
with, however, the momentum transfer due to the \( n \rightarrow s \) transitions taken into account. Thus for the normal fluid component, we assume an Euler equation, so that

\[
\frac{\partial}{\partial t} (p_n v_n) + \text{div} (p_n v_n v_n) = -\nabla p_n + \gamma v_s,
\]

(345)

where the term in \( \gamma v_s \) represents the volume rate of increase of normal fluid momentum due to transitions. For the superfluid component, we make use of the results obtained earlier in this section to write the momentum equation as

\[
\frac{\partial}{\partial t} (p_s v_s) + \text{div} (p_s v_s v_s) = -p_s \nabla \Phi_s - \omega \times \text{curl} \Lambda - \gamma v_s,
\]

or

\[
\frac{\partial}{\partial t} (p_s v_s) + \text{div} (p_s v_s v_s) = \text{div} (\Lambda \omega - p_s I) - \gamma v_s,
\]

(346)

or

\[
\frac{\partial}{\partial t} v_s + v_s \cdot \nabla v_s = -\nabla \Phi_s - \frac{\omega}{p_s} \times \text{curl} \Lambda.
\]

(the three forms are all equivalent). The equations (343) - (346) form a complete set of hydrodynamic equations for reversible processes. From these equations, we may derive the two energy equations

\[
\frac{\partial}{\partial t} \left\{ \frac{1}{2} p_n v_n^2 + p_n e_n \right\} + \text{div} \left\{ \left( \frac{1}{2} p_n v_n^2 + p_n e_n + p_n \right) v_n \right\} = \gamma (\Phi_n - \frac{1}{2} v_n^2 + v_n \cdot v_n),
\]

(347)

and

\[
\frac{\partial}{\partial t} \left\{ \frac{1}{2} p_s v_s^2 + p_s e_s \right\} + \text{div} \left\{ \left( \frac{1}{2} p_s v_s^2 + p_s e_s + p_s \right) v_s \right\} = (\Lambda \omega - p_s I) \cdot v_s + \]

...
\[ + \left( \frac{\omega}{p_s} \times \text{curl} \lambda \right) \times \lambda \right\} = - \Gamma \left( \Phi_s + \frac{1}{2} v_s^2 \right). \] (348)

The terms on the right-hand side give the rate of energy exchange due to \( f_n \leftrightarrow f_s \) transitions (as may be easily verified by a direct calculation of this energy exchange). For purposes of comparison with the results of Bekarevich and Khalatnikov, we may write the equations in the following form:

\[ \frac{\partial v_n}{\partial t} + v_n \cdot \nabla v_n = - \nabla p_x - \frac{(1-x)}{x} \nabla T - (1-x) \nabla \frac{1}{2} w^2 + \frac{\omega}{c} \nabla \lambda - \frac{\Gamma c}{c_n}, \] (349)

\[ \frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s = - \frac{\nabla p}{c} + x \frac{1}{2} w^2 + \nabla T + \frac{\omega}{c} \nabla \lambda - \frac{\omega}{c_s} \times \text{curl} \lambda, \] (350)

and

\[ \frac{\partial \lambda}{\partial t} + \text{div} \left\{ f_n v_n \nabla v_n + f_s v_s \nabla v_s + p \frac{\partial}{\partial T} - \frac{\lambda \omega \omega}{\omega} \right\} = 0. \] (351)

These equations are in agreement with the equations of Bekarevich and Khalatnikov in the special case when the coefficients \( \beta, \beta' \) and \( \gamma \) of their theory (cf. equations (298) - (300)) are all zero. In the present case, however, the term \( \frac{\omega}{c_s} \times \text{curl} \lambda \) in the superfluid equation is not a dissipative term but comes from the stress tensor of the perfect fluid theory. Furthermore, the extra term in the stress tensor \( \lambda \omega \) is here associated with the superfluid component and not with the normal fluid component as in their theory.
We now consider the extension of the present theory to include dissipative processes. As mentioned earlier, the systematic procedure used in Chapter III for deriving the equations for dissipative processes cannot be applied in a straightforward manner to the present theory (mainly because we are here interested in a volume dissipative process—mutual friction—which does not fit easily into the framework of the general theory of Chapter III). For this reason, we do not attempt to consider the most general dissipative equations; rather, we restrict attention to the type of dissipative terms considered by Bekarevich and Khalatnikov. More specifically, we consider two dissipative processes—(i) a mutual friction force effecting momentum exchange between the components and (ii) a dissipative stress tensor acting on the normal fluid component. (In the present theory, as in the theory, as in the theory of Bekarevich and Khalatnikov, we may easily include thermal conduction and the dissipative gradient terms in the superfluid equation (cf. II-B-2); however, for purposes of comparison with their equations we will omit such terms here.) The independent equations will be the conservation equations for mass, momentum and energy, and the equation for the superfluid velocity. These equations may be written as

\[
\frac{\partial \rho}{\partial t} + \text{div} \left( \rho_n v_n + \rho_s v_s \right) = 0, \tag{352}
\]

\[
\frac{\partial \mathbf{j}}{\partial t} + \text{div} \left\{ \rho_n v_n \mathbf{v}_n + \rho_s v_s \mathbf{v}_s + \rho \mathbf{v} - \lambda \frac{\mathbf{w}}{\omega} \right\} = 0, \tag{353}
\]

\[
\frac{\partial \mathbf{E}}{\partial t} + \text{div} \left\{ Q_0 + Q' \right\} = 0, \tag{354}
\]
where
\[ Q_0 = \left[ \frac{1}{2} \rho_n v_n^2 + p_n e_n + p_n \right] v_n + \left[ \frac{1}{2} p_s v_s^2 + p_s e_s \right] v_s - \left[ \frac{\lambda}{\omega} \frac{\omega}{\omega} - \rho_s \frac{I}{I} \right] v_s \]
\[ + \left[ \frac{\omega}{\xi} \times \text{curl} \Lambda \right] \times \lambda, \]  \hfill (355)

and
\[ \frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s = -\nabla \Phi_s - \frac{\omega}{\xi} \times \text{curl} \lambda + \frac{\xi}{\xi}, \]

where \( \frac{I}{I} \) is the dissipative stress tensor, \( \rho_s \xi \) is the mutual friction force per unit volume and \( Q' \) is the contribution to the energy flux from the dissipative processes. We may obtain the form of \( Q' \) by plausible arguments. First of all, there will be a term \(- \frac{I}{I} \cdot v_n \) giving the rate of working of the stress tensor. \( \frac{I}{I} \). We may expect that the term \( \frac{\xi}{\xi} \) in the superfluid equation will affect the transport of vorticity and will thus lead to an additional term in the energy flux. The vorticity equation is
\[ \frac{\partial \omega}{\partial t} = \text{curl} \left\{ (v_s + \frac{1}{\xi} \text{curl} \Lambda) \times \omega + \frac{\xi}{\xi} \right\}; \]  \hfill (356)

thus the rate of transport of vorticity across a surface with normal \( \mathbf{n} \) is
\[ -\mathbf{n} \times \left\{ (v_s + \frac{1}{\xi} \text{curl} \Lambda) \times \omega + \frac{\xi}{\xi} \right\}. \]  \hfill (357)

Then the rate of energy transport by this process in the rest frame of the superfluid is
\[-\lambda \cdot n \times \left\{ \left( \frac{1}{\rho_s} \text{curl} \ \lambda \right) \times \omega + \boldsymbol{f} \right\} = n \cdot \left\{ \lambda \times \boldsymbol{f} + \left( \frac{\omega}{\rho_s} \right) \times \text{curl} \ \lambda \right\} \times \lambda \right\}, \quad (358)\]

so that we would expect a term \(\lambda \times \boldsymbol{f}\) in the dissipative energy flux \(\mathcal{Q}'\). Thus we assume

\[\mathcal{Q}' = \lambda \times \boldsymbol{f} - \mathcal{I} \cdot \nabla n. \quad (359)\]

For the entropy, we expect an equation of the form

\[\frac{\partial}{\partial t} (\rho s) + \nabla \cdot (\rho s \nabla n) = \frac{R}{T}, \quad (360)\]

with \(R\) positive definite. Equation (360) is not independent, and, from the hydrodynamic equations (352) - (355), we may show that

\[R = \mathcal{I} : \nabla \nabla n + \frac{f}{\rho_s} \cdot \left\{ \frac{1}{\rho_s} \nabla - \text{curl} \ \lambda \right\}. \quad (361)\]

We now assume (as did Bekarevich and Khalatnikov) that the dissipative processes represented by \(\mathcal{I}\) and \(\boldsymbol{f}\) are independent in the sense that their contributions to the dissipation function are independently positive definite. Then \(\mathcal{I}\) must have the form

\[\mathcal{I}_{ik} = \mu \text{ln} \ | \mathbf{v}_n | \frac{\partial \mathbf{v}_n}{\partial x_i}; \]

if we assume that the viscosity tensor is isotropic, then this reduces to
\[ \tau_{ik} = 2 \mu_n \left\{ \epsilon_{ij}^{(n)} - \frac{1}{3} \delta_{ij} e_{kk}^{(n)} \right\} + \lambda_n e_{kk}^{(n)} \delta_{ij}. \]

The mutual friction \( f \) must be determined so that

\[ -f \cdot p \geq 0 \]

where

\[ p = \text{curl} \Lambda - \epsilon_s w. \]  \hspace{1cm} (362)

We may also express \( p \) in terms of the vorticity transport velocity of the perfect fluid theory, \( \nu_{_L}^{(o)} \), where (cf. equation (329))

\[ \nu_{_L}^{(o)} = \nu_s + \frac{1}{\epsilon_s} \text{curl} \Lambda; \]  \hspace{1cm} (363)

then

\[ p = \epsilon_s \left( \nu_{_L}^{(o)} - \nu_n \right); \]

thus the dissipative mutual friction force is closely connected with the transport of superfluid vorticity relative to the normal fluid. As in the discussion of the preceding section, little more can be said about \( f \) until we have determined what quantities \( f \) may depend on. We may note, however, that the choice

\[ f = \beta' \omega \times p + \beta \nu \times \left\{ \omega \times p \right\} - \gamma \nu \left( \omega \cdot p \right), \]

where

\[ \nu = \frac{\omega}{|\omega|} \quad \text{and} \quad \beta \geq 0, \ \gamma \geq 0, \]  \hspace{1cm} (364)

satisfies (362) and leads to exactly the equations for \( \nu_n, \nu_s \).
that were obtained by Bekarevich and Khalatnikov (equations (298) - (300)). The justification of the choice (364), though, is no easier here than in the derivation of Bekarevich and Khalatnikov. We may easily write down the most general \( f \) depending only on \( \omega \) and \( \mathbf{p} \), and satisfying (362); the result (cf. equation (293)) involves 6 scalar coefficients (one of which must be positive) and includes (364) as a special case. Even if we assume that \( f \) depends only on \( \mathbf{p} \) and \( \omega \), the resulting expression for \( f \) is still more general than (364). In the case that \( \gamma = 0 \) (which, according to Bekarevich and Khalatnikov, is the case of practical interest), we can arrive at the form (364) by using the microscopic (Onsager-Feynman) theory as a qualitative guide. In this theory (as developed by Hall and Vinen), the mechanism responsible for mutual friction is the scattering of the normal fluid excitations by the quantized vortex lines. From this picture, it is reasonable to suppose that the mutual friction force is perpendicular to the superfluid vorticity \( \omega \); it is also reasonable to suppose that the relevant relative velocity is \( \mathbf{v}_n - \mathbf{v}_l^{(\omega)} \), rather than \( \mathbf{v}_n - \mathbf{v}_l \). Thus we assume that \( f \) depends only on \( \omega \) and \( \mathbf{p} = \rho \mathbf{s} (\mathbf{v}_l^{(\omega)} - \mathbf{v}_n) \) (and the scalar thermodynamic variables), and that \( f \cdot \omega = 0 \). Then it is easy to show that the most general \( f \) (also satisfying (362)) is given by

\[
f = \beta' \omega \times \mathbf{p} + \beta \mathbf{v} \times \{ \omega \times \mathbf{p} \},
\]

where

\[
\beta \geq 0,
\]
so that in this case, the present equations for $v_n$ and $v_s$ agree exactly with those of Bekarevich and Khalatnikov in the case $Y = 0$.

This completes the alternative derivation of the equations of Bekarevich and Khalatnikov. Although the final equations for $v_n$ and $v_s$ are the same in both derivations, it is hoped that the present derivation brings out more clearly the physical significance of the various terms and also the nature of the arguments needed to obtain the final equations.
3. Physical basis of the theory

As mentioned earlier, the theory of Bekarevich and Khalatnikov is based on the two-fluid model, the usual conservation laws and invariance principles and the single additional assumption that the thermodynamic internal energy depends on the superfluid vorticity, as well as the usual thermodynamic variables of the two-fluid theory. It is this last assumption which is the new feature of their theory, and which deserves careful discussion. In the preceding two sections, we have considered the mathematical development of their theory. In the present section, we make some attempt to discuss the physical basis of the theory; in particular we wish to (i) examine the theory from the point of view of the Onsager-Feynman theory of quantized vortex lines and (ii) discuss the possibility that the theory of Bekarevich and Khalatnikov might be relevant to some microscopic picture other than the Onsager-Feynman theory. We consider (i) first.

According to the Onsager-Feynman theory the rotation of helium II gives rise to line singularities—quantized vortex lines—in the superfluid component. It is supposed that these vortex lines consist of a small core around which the circulation of the macroscopic velocity field \( \mathbf{v}_s \) is quantized according to the formula

\[
\oint \mathbf{v}_s \cdot d\mathbf{s} = \frac{2\pi n \hbar}{m}.
\]
The effective core diameter is believed to be of the order of $10^{-8}$ cm. (there is, however, some evidence that it may be much larger[37]). Thus, according to the prevailing theories, the vortex line is an "excitation" which is in some respects microscopic (the very small core) and in some respects macroscopic (because of the associated macroscopic velocity field). In the development of a hydrodynamic theory from these ideas, it is clearly desirable to try and determine to what extent the vortex lines are to be treated as microscopic excitations and to what extent they are be treated as hydrodynamic structures. We may note first of all that in a typical flow problem (e.g., the steady flow in a cylinder rotating with an angular velocity of 1 rad/sec.), the average distance between vortex lines is of the order of $10^{-2}$ cm. Thus even though the vortices are fairly close together, their average separation is still much greater than any (presently known) microscopic length scales. The energy per unit length of a vortex line is usually taken to be

$$
\pi \rho \frac{h^2}{m^2} \ln \frac{R}{a},
$$

where $R$ is of the order of the distance between vortices and $a$ is the effective core radius. This energy (which is obtained by simply integrating the kinetic energy density of a hydrodynamic vortex field) is the energy which Bekarevich and Khalatnikov have included in the thermodynamic internal energy. However, this extra energy is associated primarily
with the organized, macroscopic motion induced by the vortex; thus we may ask why it is not already included in the ordinary kinetic energy of motion. To be sure, if we consider a spatial average, \( <v_s> \), of the superfluid velocity field over a region of space containing many vortex lines, then, for

\[
v_s = <v_s> + v_s'
\]

\[
\left< \frac{1}{2} \rho_s v_s^2 \right> = \frac{1}{2} \rho_s <v_s>^2 + \frac{1}{2} \rho_s <v_s'^2> \neq \frac{1}{2} \rho_s <v_s>^2
\]

however, in this case, the extra energy is analogous to the extra energy of turbulent fluctuations in the turbulent motion of an ordinary fluid—it is in no way associated with the thermodynamic internal energy, since it appears as a consequence of averaging a macroscopic velocity field and not as a consequence of averaging over a thermal distribution. The essence of the argument here is that the vortex motion will appear regular and macroscopic in character when viewed on a sufficiently small \( \sim 10^{-2} \text{ cm} \)—but still macroscopic—length scale. One would expect some sort of equilibrium thermodynamics to hold for volume elements whose linear extent is even smaller than \( 10^{-2} \text{ cm} \); thus it is not clear that the continuum model of Bekarevich and Khalatnikov is relevant to the Onsager-Feynman theory of quantized vortex lines. The turbulence analogy, however, does suggest an alternative approach to constructing a hydrodynamic theory on the basis of the two-fluid model and the Onsager-Feynman theory. This will be considered in detail in the next section (IV-C).
We may also consider the possibility that the theory of Bekarevich and Khalatnikov is an adequate continuum representation of some microscopic theory other than the Onsager-Feynman theory. To discuss this we may first ask for a description in general terms of the physical picture associated with the inclusion of a vorticity dependent term in the internal energy function. The presence of an extra term dependent on the vorticity in the energy means that there is some internal structure in the fluid which tends to resist rotation (it is not really necessary to assume that this structuring is produced by the rotation; as Lin* has pointed out, the excess energy may be that which is required to align structures already present in the fluid). The fact that this extra energy is included in the thermodynamic internal energy would seem to correspond to a situation in which the internal structures are microscopic in size (e.g., as Lin[25] has suggested, the structures may be small vortex rings of sub-macroscopic diameter). Thus we have a plausible, general sort of microscopic picture to which the theory of Bekarevich and Khalatnikov would seem to be applicable. However, there is a further point regarding the development of such a theory which we discuss by means of an example—namely, we consider the development of the hydrodynamic equations for an ordinary fluid in which there is an internal angular momentum in addition to the usual $\mathbf{r} \times \mathbf{v}$ angular momentum density (the extra angular momentum is associated with the rotation of the constituent molecules of the fluid about their centers of mass; this theory * private communication
has been developed systematically by Grad [10], and a concise presentation is given in [4, p.304]). The essential features of this theory relevant to the discussion here are (i) there is an extra term in the energy dependent on a local internal angular velocity $\Omega$, $\Omega$ being the local mean angular velocity of the molecules, (ii) the rate of energy dissipation from the processes which tend to produce rotational equilibrium is proportional to the square of the difference $\Omega - \frac{1}{2} \text{curl} \mathbf{v}$, and (iii) if the relaxation time for these processes is much less than the relevant macroscopic time scales, then $\Omega \overset{\sim}{\sim} \frac{1}{2} \text{curl} \mathbf{v}$. Thus in the case of helium II, we might expect that there would be some internal parameter such as $\Omega$ above to describe the mean state of the internal structures; then the extra energy would depend on $\Omega$ and the dissipation would depend on $\Omega - \frac{1}{2} \text{curl} \mathbf{v}$.

For steady flows (or flows with slow time variations) we would have $\Omega \overset{\sim}{\sim} \frac{1}{2} \text{curl} \mathbf{v}$; thus the quantity $\omega \equiv \text{curl} \mathbf{v}$ would enter into the thermodynamics through a sort of equilibrium condition and not as a fundamental thermodynamic variable. The detailed development of a theory along the lines described here may well differ considerably from the development given by Bekarevich and Khalatnikov for their theory.

In summary: there are definite arguments which indicate that the theory of Bekarevich and Khalatnikov may not be relevant to the two-fluid model with quantized vortex lines; although their theory may have some relevance to other microscopic pictures, it is possible that the detailed development of such theories should proceed along somewhat different lines.
C. An Alternative Approach to the Hydrodynamic Theory with Quantized Vortex Lines

In the theory of Bekarevich and Khalatnikov, the hydrodynamic equations obtained are a generalization of Landau's equations (as discussed in III-B); also, their equations are (ostensibly) equations for local macroscopic variables (\( \mathbf{v}_s \), \( \mathbf{v}_n \) etc.). Hall and Vinen\[12,13,14,38\] have taken an alternative approach—namely, they attempt to obtain equations for averages over regions containing many vortex lines of the local variables. Thus in their theory, the basic equations for the local quantities are essentially the same as the Landau equations; however, the particular solutions of interest—flows involving a large number of vortex lines—are too complicated to calculate in detail, so that the development of equations for average quantities is essential. In carrying out this development of the equations for average quantities, Hall and Vinen find it necessary to make a number of additional specific assumptions about the nature of the vortex flows. In view of our present uncertain knowledge regarding the detailed structure and properties of quantized vortex lines, it seems desirable to base the derivation of the hydrodynamic equations on as few assumptions about the specific features of vortex flows as possible. In the present section, then, we offer an alternative derivation of the equation proposed by Hall \[13,14\] to describe "vortex waves". We consider only the case in which mutual friction is absent (or unimportant; as in the case for lower temperatures), mainly because it is not clear how to include mutual friction in the present
theory. (In this connection, however, we may note that the theory of mutual friction given by Hall and Vinen is not free of conceptual difficulties. For example (i) it is not clear why the Magnus effect should be relevant, since one would expect the vortices to move with the local superfluid velocity, (ii) they do not distinguish clearly between mean and local variables in their derivation, and (iii) in their treatment of roton-vortex line scattering, they (apparently) regard the force as acting on the vortex core, rather than being distributed throughout the fluid; however, the cross-section is calculated from an interaction term of the form $\mathbf{P} \cdot \mathbf{v}_s$, where $\mathbf{P}$ is the roton momentum, and $\mathbf{v}_s$ is the macroscopic vortex velocity field.)

For now, we wish only to consider those flows for which the detailed structure of the vortex core is unimportant. Thus flows in which the processes of vortex line creation and destruction and mutual friction are important are outside the scope of the present discussion. We are then considering only the hydrodynamic aspects of the vortex lines, and we may thus expect that the continuum hydrodynamic equations will be adequate for describing the flow. We are still assuming that the superfluid component is pointwise irrotational, so that the appropriate hydrodynamic equations are the Landau equations as discussed in II-A (we will not be concerned with dissipative processes in the present discussion, so we use the simpler perfect fluid equations of Chapter II). The equation for $\mathbf{v}_s$ may be written as (cf. equation (33))
\[
\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s = -\nabla \phi \tag{366}
\]

(where \( \nabla \phi = \nabla \mathbf{v}_s - \frac{\nabla P}{\rho} + \nabla \frac{1}{2} \mathbf{w}^2 \)).

In the case of moderate velocities and small variations of temperature in the flow, we also have

\[
\text{div} \ \mathbf{v}_s = 0. \tag{367}
\]

Equations (366) and (367) are formally the same as the equations for ordinary incompressible perfect fluids. Thus possible solutions are flows in which the vorticity is concentrated in linear singularities, and we are interested in solutions of this type where the number of vortex lines is large. (In accordance with the quantized vortex line theory, we are assuming here that the response of the superfluid to rotation is an array of vortex lines.) Here, as in the case of the turbulent flow of an ordinary fluid, we may expect that the detailed velocity distribution is irregular and fluctuating, even when the external conditions are steady. Since it is obviously not practicable to attempt a detailed calculation of such a velocity field, we wish to consider some sort of average velocity. Thus we assume that the motion can be separated into a mean flow \( \langle \mathbf{v}_s \rangle \) and a fluctuating part \( \mathbf{v}'_s \), and we assume that the averaging operation commutes with space and time differentiation (in some special flows, \( \langle \mathbf{v}_s \rangle \) may be regarded as a space or time average; in more general flows,
we may consider \( \langle v_s \rangle \) to be a statistical average. Then, upon averaging, we obtain from (366) and (367) the equations

\[
\frac{\partial \langle v_s \rangle}{\partial t} + \langle v_s \rangle \cdot \nabla \langle v_s \rangle = -\nabla \langle \Phi \rangle - \text{div} \{ \langle v_s', v_s' \rangle \}
\]

(368)

and

\[
\text{div} \langle v_s \rangle = 0.
\]

We may note first of all that \( \text{curl} \langle v_s \rangle \neq 0 \) in general; in fact, if \( N(c) \) denotes the number of vortex lines passing through a circuit \( C \), we have

\[
\oint_C \langle v_s \rangle \cdot ds = N(c) \cdot \frac{h}{m}
\]

so that

\[
\oint_C \langle v_s \rangle \cdot ds = \frac{h}{m} \langle N(c) \rangle
\]

and

\[
\iint_A \text{curl} \langle v_s \rangle \cdot n \, d\sigma = \frac{h}{m} \langle N(c) \rangle.
\]

(370)

We now consider the correlation term \( \langle v_s', v_s' \rangle \) appearing in equations (368) for the mean flow. This term is analogous to the Reynolds' stress in the equations for the turbulent flow of an ordinary fluid. However, the detailed structure of the vortex flows under consideration here would seem to bear little resemblance to the structure of turbulent flow. What is needed, in fact, is a statistical theory of vortex line flows; such a theory has not been developed, however, and, at present, we can only make a few conjectures about the nature of these flows. From the point of view of applications, it
would be very desirable to express the correlation $\langle \mathbf{y}' \mathbf{y}' \rangle$ in terms of mean flow quantities. If the vortex line array is very chaotic and if there is little correlation between the directions of neighboring filaments*, then, on the basis of the ordinary theory of turbulence, we might be skeptical about the possibility of relating the correlations to the mean flow in a universal manner. However, for more regular vortex flows, we might hope to relate $\langle \mathbf{y}' \mathbf{y}' \rangle$ to the mean flow in some simple manner. (An example of what we mean by a regular vortex flow" is perhaps helpful here. Hall and Vinen have conjectured that, for a helium filled circular cylinder in steady rotation, the vortex lines are parallel to the axis of rotation, the array is (nearly) uniform in spacing, and the lines are at rest when viewed in a reference frame rotating with the cylinder. In the present discussion, we would only conjecture that, on the average, the array has these properties; it seems likely that at any given instant there would be small, irregular deviations.) If we suppose that we are dealing with a regular vortex flow, in the sense described above, the neighboring vortex filaments will be approximately parallel, and will have the direction of the mean vorticity, \( \text{curl} \langle \mathbf{y} \rangle \). Roughly speaking then, the averaging implied in $\langle \mathbf{y} \rangle$ will in this case be over the possible positions—and not the possible directions—of the vortex filaments. Since the velocity fields induced by the vortices are transverse to the filament direction, we make the plausible assumption that the correlation tensor $R_{ij} = \langle \mathbf{y}_i \mathbf{y}_j' \rangle$ is transverse to $\omega = \text{curl} \langle \mathbf{y} \rangle$.* Vinen [38] has considered such flows; he uses the terminology "superfluid turbulence" to describe such a flow.
that is, we assume that

\[ \omega_i R_{ij} = 0. \]  \hspace{1cm} (371)

If \( R_{ij} \) is to satisfy (371), it must depend on \( \omega_i \). The simplest case is when \( R_{ij} \) depends only on \( \omega_i \) (and possibly scalar quantities such as \( \rho_\sigma \), etc.); then \( R_{ij} \) must have the form

\[ R_{ij} = \lambda \left\{ \omega^2 \delta_i^j - \omega_i \omega_j \right\}, \]  \hspace{1cm} (372)

and we assume this form here. We may relate the scalar \( \lambda \) to \( \epsilon \), the energy per unit length of a vortex line, as follows: consider a vortex tube (i.e., a vortex tube defined by the mean vorticity - curl \( \langle \omega_\sigma \rangle \)) of length \( dJ \), cross-section \( dA \); the energy in this tube associated with the fluctuations is

\[ \frac{1}{2} \rho_\sigma R_{ij} \cdot dA dJ = \frac{1}{2} \rho_\sigma 2\lambda \omega^2 dA dJ; \]

since the mean total length of vortex line in the tube is \( \frac{dJ \cdot \omega dA}{K} \) (where \( K = \frac{h}{\omega} \) is the circulation around a single vortex), this energy may also be written as

\[ \epsilon dJ dA \cdot \frac{\omega}{K}, \]

so that we obtain

\[ \lambda = \frac{\epsilon}{\rho_\sigma \omega K}. \]  \hspace{1cm} (373)

If we introduce \( \nu = \epsilon / \rho_\sigma \omega K \), then the equation for \( \langle \omega_\sigma \rangle \) may be written as
\[ \frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \cdot \nabla \langle \mathbf{v} \rangle = - \nabla \left\{ \mathbf{\Phi} + \nu \omega \right\} + \text{div} \left\{ \frac{\nu \omega^2}{\omega} \right\} \]  \quad (374)

and this is the equation given by Hall [13] to describe vortex waves.

At present, the above theory must be regarded as simply a conjecture; however, it is at least free of specific assumptions concerning the nature of the vortex flows (such as Hall and Vinen's assumption that the Magnus effect is relevant). As mentioned earlier, what is needed is a systematic development of a statistical theory of flows containing a large number of vortex lines, and the justification of the conjectures here must await the development of such a theory.