CHAPTER V - SUMMARY

A. Introduction

In the preceding chapters we have considered several theories of the hydrodynamics of liquid helium. At present, the experimental evidence is not sufficient to choose among (or reject all of) these theories. A conservative but definite statement concerning the comparison of theory with experiment is that (i) the Landau equations for reversible flows have been well verified experimentally for flows in which dissipative effects are unimportant and (ii) the Landau equations are inadequate to describe all of the experimental results. The central problem in the hydrodynamics of helium II is to determine how the Landau equations are to be modified in order to describe dissipative flows, and the theories discussed in the preceding chapters represent some of the current approaches to this problem. In this chapter, we present a summary of these theories in the form of a unified mathematical scheme which includes the theories discussed earlier as special cases.

In part B, the general thermodynamic and hydrodynamic equations applicable to all cases are given. In part C, the various special cases are discussed with the aid of the general framework of part B.
B. General Equations

1. Thermodynamics

In the discussion of the general equations, it is convenient to use the separate thermodynamic description for each component. Since this description has been discussed in detail in sections II-A-2 and IV-B-2, we only summarize the relevant formulae here. It is supposed that the total energy per unit volume \( E \) may be split into two parts—the superfluid part \( E_s \) and the normal fluid part \( E_n \). Thus

\[
E = E_n + E_s
\]

where

\[
E_n = \frac{1}{2} c_n v_n^2 + \rho_n \mathcal{E}_n
\]

and

\[
E_s = \frac{1}{2} c_s v_s^2 + \rho_s \mathcal{E}_s
\]

The specific internal energy \( E_n \) is supposed to be a function of \( \rho_n \) and \( S_n \) (\( S_n \) being the entropy per unit mass of normal fluid), and the specific internal energy \( E_s \) is supposed to be a function of \( \rho_s \) and \( \omega \equiv |\text{curl} \, \mathbf{v}_s| \). The partial pressures \( P_n \), \( P_s \) and the temperature \( T \) are defined by

\[
d\rho_n = \frac{P_n}{\rho_n^2} \, d\mathcal{E}_n + T \, dS_n, \tag{376}
\]

and

\[
d\mathcal{E}_s = \frac{P_s}{\rho_s^2} \, d\mathcal{E}_s + \lambda \, d \left( \frac{\mathcal{E}_s}{\rho_s} \right).
\]

The condition for equilibrium with respect to \( E_n \leftrightarrow E_s \) transitions
is given by

\[ \Phi_n - \Phi_s = \frac{1}{2} w^2, \quad (w = v_n - v_s) \]

where

\[ \Phi_n = e_n + \frac{p_n}{c_n} - TS_n \]  \hspace{1cm} (377)

and

\[ \Phi_s = e_s + \frac{p_s}{c_s} - \frac{\lambda \omega}{c_s} . \]

The thermodynamic quantities for the composite system are then given by

\[ e = x e_n + (1-x)e_s , \]

\[ s = x s_n , \quad p = p_n + p_s , \]  \hspace{1cm} (378)

and the differential of \( e \) is given by

\[ de = \frac{p}{c_s^2} dp + T ds + \lambda d\left( \frac{\omega}{c_s} \right) + \frac{1}{2} w^2 dx . \]  \hspace{1cm} (379)

The differentials of the partial pressures may be expressed in terms of the thermodynamic variables of the composite system as follows:

\[ \frac{dp_n}{c_n} = \frac{dp}{c} + \frac{(1-x)}{x} s dT + (1-x) \frac{1}{2} w^2 - (1-x) \frac{\omega}{c_s} d\lambda , \]

and

\[ \frac{dp_s}{c_s} = \frac{dp}{c} - s dT - x \frac{1}{2} w^2 + x \frac{\omega}{c_s} d\lambda . \]  \hspace{1cm} (380)

In the case that the internal energy does not depend on \( \omega \),
we simply take \( \lambda = 0 \) in the above formulae.

It should be pointed out that the splitting of the energy by (375) and the thermodynamic formulae (376) are really additional assumptions concerning the nature of the two-fluid model. (Although these assumptions are consistent with the view of helium II as a "mixture" of two interpenetrating fluids which can, in the first approximation, move freely through one another, it is not clear at present that we have the right to ascribe an independent existence to each fluid to the extent of giving a separate thermodynamic description for each).
2. Hydrodynamic equations

The basic equations for reversible flows of helium II (in the case that the energy does not depend on the superfluid vorticity \( \lambda = 0 \)) are the Landau equations. In more general theories, these equations are supplemented by some combination of the following: (i) additional stress tensors \( \tau^{(n)} \), \( \tau^{(s)} \) acting on the normal and supercomponents respectively, (ii) a mutual friction force per unit volume, \( F \), (iii) dependence of the thermodynamic internal energy on the superfluid vorticity \( \omega = |\text{curl} \mathbf{v}_s| \), (iv) an additional term in the entropy flux, \( \mathbf{j}_k \), and (v) an additional term in the energy flux, \( \mathbf{Q}' \). The general hydrodynamic equations may then be written in the following form:

**Mass:**

- **normal fluid**: \( \frac{\partial \rho_n}{\partial t} + \text{div} (\rho_n \mathbf{v}_n) = \Gamma \) (where \( \Gamma \) is the volume rate of conversion, and is determined by the flow conditions and the equilibrium condition (577).)

- **superfluid**: \( \frac{\partial \rho_s}{\partial t} + \text{div} (\rho_s \mathbf{v}_s) = -\Gamma \)

**total**: \( \frac{\partial \rho}{\partial t} + \text{div} (\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s) = 0 \) (381)

**Momentum:**

- **normal fluid**: \( \frac{\partial}{\partial t} (\rho_n \mathbf{v}_n) + \text{div} (\rho_n \mathbf{v}_n \mathbf{v}_n) + \nabla P_n - \Gamma \mathbf{v}_s = \text{div} \tau^{(n)} - F \) (382)

- **superfluid**: \( \frac{\partial}{\partial t} (\rho_s \mathbf{v}_s) + \text{div} (\rho_s \mathbf{v}_s \mathbf{v}_s) + \nabla P_s + \Gamma \mathbf{v}_s = \text{div} \left\{ \frac{\lambda}{\omega} \mathbf{v}_s \right\} + \text{div} \tau^{(s)} + F \) (383)

**total**: \( \frac{\partial}{\partial t} (\rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s) + \text{div} (\rho_n \mathbf{v}_n \mathbf{v}_n + \rho_s \mathbf{v}_s \mathbf{v}_s) + \nabla P = \text{div} \left\{ \tau^{(n)} - \frac{\lambda}{\omega} \mathbf{v}_s \right\} + \text{div} \tau^{(s)} + F \) (384)

where for reasons which will become apparent, we have chosen

*Corresponding to ordinary thermal conduction.*
to write the total additional stress tensor in the supercompo-
net in the form \( \frac{\lambda}{\omega} \omega \omega + T^{(\omega)} \). The equation for the super-
fluid vorticity may be written as

\[
\text{vorticity: } \frac{\partial \omega}{\partial t} = \text{curl} (\nu_\perp \times \omega) + \text{curl} f,
\]

where
\[
\nu_\perp = \nu_s + \frac{1}{\rho_s} \text{curl} \lambda \nu,
\]
and
\[
f = \frac{E}{\rho_s} + \frac{1}{\rho_s} \text{div} \gamma \omega.
\]

(385)
(386)
(387)

The scalar \( \omega \) satisfies the equation

\[
\text{scalar vorticity: } \frac{\partial \omega}{\partial t} + \text{div}[\nu_\perp \omega + \nu_\times f] = \omega \nu_\perp \cdot \text{grad} \nu_\perp + f \cdot \text{curl} \nu.
\]

(388)

The energy equation is

\[
\text{energy: } \frac{\partial}{\partial t} \left\{ E_n + E_s \right\} + \text{div} \left( Q_0 \right) = -\text{div} Q',
\]

(389)

where \( E_n, E_s \) are the total energies per unit volume of the
normal and supercomponents (cf. (375)), and where \( Q_0 \), the
energy flux vector of the Landau equations, is given by

\[
Q_0 = (E_n + P_n) \nu_n + (E_s + P_s) \nu_s.
\]

(390)

The hydrodynamic equations must imply an equation for the en-
tropy of the form

\[
\text{entropy: } \frac{\partial}{\partial t} (E_s) + \text{div}(E_s \nu_n) = -\text{div} \left\{ \frac{E_s}{\rho_s} \right\} + \frac{R}{\rho}.
\]

(391)
where $R$, the (local) volume rate of energy dissipation, is to be positive definite. From the equations (381)-(391) one may show that

$$
R = \tau^{(m)} : \text{grad} \eta + \tau^{(e)} : \text{grad} \rho + \mathbf{F} \cdot (\nu_n - \nu_s) - \mathbf{f}_s \cdot \nabla T - \\
- \text{div} \left\{ \mathbf{Q}' - \left[ \tau^{(m)} : \nu_n - \left( \frac{\lambda}{\omega} \omega \omega + \tau^{(e)} \right) \cdot \nu_s \right] + \lambda \omega (\nu_s - \nu_s) + \lambda \omega \left( \frac{\nu_s \cdot \mathbf{f}}{\omega} \right) + \mathbf{T} \mathbf{f}_s \right\}.
$$

(392)

From (392), it is plausible to take

$$
R = \tau^{(m)} : \text{grad} \eta + \tau^{(e)} : \text{grad} \rho + \mathbf{F} \cdot (\nu_n - \nu_s) - \mathbf{f}_s \cdot \nabla T,
$$

(393)

and

$$
\mathbf{Q}' = - \frac{\tau^{(m)}}{\omega} \nu_n - \left( \frac{\lambda}{\omega} \omega \omega + \tau^{(e)} \right) \cdot \nu_s + \lambda \omega (\nu_s - \nu_s) + \lambda \omega \left( \frac{\nu_s \cdot \mathbf{f}}{\omega} \right) + \mathbf{T} \mathbf{f}_s.
$$

(394)

In justification of the splitting of (392) into (393) and (394), we may note that (i) the equations (393) and (394) give the correct expressions for $R$ and $\mathbf{Q}'$ for all of the various theories discussed in the previous chapters and (ii) it is possible to give a physical interpretation of the various terms in the expression (394) for $\mathbf{Q}'$. Of course, the fact that $R$ must be positive definite—in conjunction with (392)—severely limits the possible forms of $R$ and $\mathbf{Q}'$. In particular, in the case $\lambda = 0$ and $\mathbf{F} = 0$, the splitting of (392) into (393) and (394) may be proved under certain mild conditions on $\tau^{(m)}$, $\tau^{(e)}$, and $\mathbf{f}_s$ (cf. Appendix to Chapter III). In the general case,
however, we must rely on physical interpretation to assure us that \( Q' \) as given by (394) is the correct expression for the additional energy flux, and, hence, that (393) is the correct expression for the dissipation function.

The reason for writing the total additional superfluid stress tensor as \( \frac{\lambda \omega \omega}{\omega} + \tau^{(s)} \) is clear from (393): only the part \( \tau^{(s)} \) is a dissipative stress.

The hydrodynamic equations given above provide a general framework for a discussion of the various theories, and this will be taken up in the next section. Although the question of boundary conditions will be discussed in conjunction with each particular case in the next section, there is one general point regarding boundary conditions which we wish to make here. Suppose we consider a helium II--solid boundary; let \( Q |_{\text{rest frame}} \) denote the total energy flux in the helium, evaluated in the rest frame of the solid. Then let

\[
Q |_{\text{rest frame}} = Q_1 + \tau \mathbf{\tau} \tag{395}
\]

that is, \( Q_1 \) is everything in the energy flux except the entropy flux term. In the solid, there will be a heat current \( \mathbf{H} \), and an associated entropy flux \( \mathbf{H} / \tau \). The conservation of energy requires that

\[
Q_1 \cdot \mathbf{n} + \tau \mathbf{\tau} \cdot \mathbf{n} = \mathbf{H} \cdot \mathbf{n} \tag{396}
\]

where, for definiteness, we take \( \mathbf{n} \) to be the unit normal pointing into the fluid. If \( Q_1 \cdot \mathbf{n} = 0 \), then the entropy flux is continuous; if \( Q_1 \cdot \mathbf{n} \neq 0 \), however, then the entropy flux is
not continuous, and the interface acts (formally) as an entropy source—that is, there is extensive dissipation at the boundary. The net production of entropy at the interface must be positive; the entropy flowing away from the interface into the fluid is $\mathcal{E}_f \cdot n$, and the entropy flowing away from the interface into the solid is $-\mathcal{E}_s \cdot n / \tau$, thus we must require

$$\mathcal{E}_f \cdot n - \frac{\mathcal{E}_s \cdot n}{\tau} \geq 0$$

or

$$Q \cdot n \leq 0.$$  \hspace{1cm} (397)

Thus whatever the boundary conditions imposed, they must be such that (397) is satisfied. In the case of an ordinary fluid (with the usual no-slip boundary conditions), the quantity $Q \cdot n = 0$; in several of the hydrodynamic theories of helium however, the quantity $Q \cdot n$ does not vanish, and the restriction (397) must be satisfied (and, in fact, in some of the theories, one may use the condition (397) to determine an appropriate boundary condition). Further discussion of this point will be given in the next section in connection with each particular case. (In the above analysis, we have tacitly assumed that the temperature is continuous at the interface. If we include the possibility of a temperature jump (Kapitza effect), then (397) must be replaced by

$$Q \cdot n \leq \frac{T_{wall} - T_{ne}}{T_{wall}} \mathcal{E}_s \cdot n ;$$

according to the Kapitza effect relation, $T_{wall} - T_{ne} = A_k (\mathcal{E}_s \cdot n)$.
where $A_k > 0$ is the thermal resistance of the boundary, and the condition becomes

$$A_k \frac{(H \cdot n)^2}{T_{\text{wall}}} - Q \cdot n \geq 0.$$  

If we assume that the Kapitza effect and the dissipative effects included in $Q_i$ contribute independently to the dissipation, then we again require $Q \cdot n < 0$; in particular, this will be true if the boundary conditions are such that $Q \cdot n$ is independent of the magnitude of the heat current, $H \cdot n$.}
C. Equations for Particular Theories

1. Reversible flows in the case $\lambda=0$.

In the case that the internal energy doesn't depend on the superfluid vorticity, the Landau equations are recovered for $\zeta^{(m)}$, $\zeta^{(s)}$, $\mathcal{F}$ and $\mathcal{J}_s$ all zero. In the present general scheme, the Landau equations appear as only a possible set of perfect fluid equations. However, one may deduce the Landau equations from the conservation laws and the requirement that $\text{curl } \mathbf{v}_S \equiv 0$ (cf. II-A-1; we may also note that one may deduce the Landau equations from the conservation laws and the less stringent requirement that the superfluid vorticity must move with the superfluid velocity—-that is, that $\frac{\partial}{\partial t} \text{curl } \mathbf{v}_S = \text{curl} \{ \mathbf{v}_s \times \text{curl } \mathbf{v}_S \}$). An alternative approach is given in section II-A-2, where it is shown that one may obtain the Landau equations (without the restriction $\text{curl } \mathbf{v}_S = 0$) from the assumption that the two components are two independent perfect fluids coupled only by the $\mathcal{E}_{n} \leftrightarrow \mathcal{E}_{s}$ transitions, with the transitions taking place at constant $\mathbf{v}_S$. However, these two deductions of the Landau equations depend on assumptions whose validity (even in the perfect fluid theory) is not evident. In particular, we see from equation (393) for the dissipation function, that a mutual friction force $\mathcal{F}$ which is perpendicular to $\mathbf{w} = \mathbf{v}_n - \mathbf{v}_s$ does not lead to any dissipation ($\mathbf{v}_{n} \equiv \mathbf{v}_s$ in the case $\lambda=0$) and may thus be included in the perfect fluid equations. Indeed, the equations proposed by Lin (cf. II-B-4) differ from the Landau equations by just a term (namely, $\mathcal{F} = \mathbf{e}_x (1-x) \mathbf{w} \times \text{curl } \mathbf{v}_S$).
In general, then the Landau equations may be supplemented by a (non-dissipative) mutual friction force perpendicular to \( \mathbf{w} \). The boundary conditions to be satisfied at a wall moving with velocity \( \mathbf{U} \) are simply that \( \mathbf{n} \cdot (\mathbf{v}_n - \mathbf{U}) = 0 \) and \( \mathbf{n} \cdot (\mathbf{v}_s - \mathbf{U}) = 0 \), where \( \mathbf{n} \) is the normal to the wall.

It is natural to ask if the addition of a mutual friction force perpendicular to \( \mathbf{w} \) is the most general modification of the Landau equations allowed within the framework of the perfect fluid theory. The answer is yes, provided it is assumed that (i) the stress tensors \( \tau^{(m)} \), \( \tau^{(s)} \) are independent of the gradients of the macroscopic quantities, and (ii) the terms \( \nabla \mathbf{v}_n + \tau^{(s)} \mathbf{n} \cdot \nabla \mathbf{v}_s \) and \( \mathbf{F} \cdot \mathbf{n} \) in the dissipation function are separately zero. The assumption (i) is a reasonable assumption in a perfect fluid theory, and the assumption (ii) (corresponding to the physical statement that the volume momentum transfer processes represented by \( \mathbf{F} \) are independent of the processes represented by \( \nabla \mathbf{v}_n + \tau^{(s)} \mathbf{n} \cdot \nabla \mathbf{v}_s \)) excludes the possibility of stress tensors \( \tau^{(m)} \), \( \tau^{(s)} \) depending on \( \mathbf{w} \) and a related mutual friction \( \mathbf{F} \) specified in such a way that \( \mathbf{R} = 0 \) (cf. eq. (50), II-A-1, for an example).
2. Dissipative flows in the case $\lambda = 0$

The equations for dissipative processes are obtained by determining the quantities $\tau^{(m)}$, $\tau^{(s)}$, $\bar{F}$, and $\mathcal{F}_s$ so that the dissipation function (cf. (393)) is positive definite. Under the reasonable assumption that the contributions of $\bar{F}$ and the group $\{\tau^{(m)}, \tau^{(s)}, \mathcal{F}_s\}$ to the dissipation function are independently positive definite, one may study separately the transport terms $\tau^{(m)}$, $\tau^{(s)}$, $\mathcal{F}_s$ and the mutual friction $\bar{F}$.

We consider the quantities $\tau^{(m)}$, $\tau^{(s)}$ and $\mathcal{F}_s$ first. In the simplest case, these quantities are taken to be linear functions of the rate of strain tensors $\varepsilon_{ij}^{(m)}$, $\varepsilon_{ij}^{(s)}$ and the temperature gradient $\nabla T$, and it is assumed that the helium II is isotropic with respect to the transport processes; then the resulting equations are those of Lin's theory (III-C-2)—that is, $\mathcal{F}_s = -\frac{K}{\gamma} \nabla T$ (thermal conduction) and the stress-rate of strain relations are characterized by four shear exchange coefficients (and also four "bulk viscosity" coefficients in the general case). There are several ways in which the theory may be generalized. First, we may note that the conservation of angular momentum requires only that the sum $\tau^{(m)} + \tau^{(s)}$ be symmetric; thus $\tau^{(m)}$, $\tau^{(s)}$ could have anti-symmetric parts adding to zero, and the effect of including such terms has been considered in Chapter III (III-C-2). The effective body force associated with the anti-symmetric stress is proportional to $\text{curl} (\nabla \tau - \nabla \mathcal{F}_s)$, and there is a (dissipative) volume exchange of angular momentum between the two components induced by the
the stress. Another way in which the theory may be generalized is to take into account the possibility that, because of the relative motion of the two fluids, the transport processes may not be isotropic (or, to put it another way, the viscosity tensors and the thermal conductivity tensor may depend on the direction of the relative velocity \( \mathbf{w} \)); the resulting general equations are very complicated and have not been studied in detail.

The mutual friction \( \mathbf{F} \) must be such that \( \mathbf{w} \cdot \mathbf{F} \geq 0 \), and we can say little more about \( \mathbf{F} \) until we know what vectors \( \mathbf{F} \) may depend on. In the simplest case, \( \mathbf{F} \) depends on \( \mathbf{w} \) alone, and we have \( \mathbf{F} = C_0 \mathbf{w} \), (where \( C_0 > 0 \) may depend on \( \mathbf{w}^2 \) as well as the scalar thermodynamic variables) and this includes the mutual friction originally proposed by Gorter and Mellink \( [2] \) in 1949. Another case of interest is when \( \mathbf{F} \) depends on \( \mathbf{w} \) and \( \mathbf{\omega} = \text{curl} \, \mathbf{v}_S \); then the most general form is \( \mathbf{F} = C_0 \mathbf{w} + C_1 \mathbf{w} \times \mathbf{\omega} + C_2 \mathbf{w} \times \left[ \mathbf{w} \times (\mathbf{\omega} \times \mathbf{w}) \right] \), (where \( C_0 > 0 \) and \( C_0, C_1, C_2 \) may depend on the scalars \( \mathbf{w}^2, \mathbf{\omega}^2 \) and \( (\mathbf{w} \cdot \mathbf{\omega})^2 \)) and this includes both the Gorter-Mellink mutual friction and the mutual friction proposed by Hall and Vinen \( [12] \) (in the case of "straight vortex lines") as special cases.

The boundary conditions to be satisfied are of some interest. When there is a heat current between a bounding wall and the helium II, the condition \( \mathbf{e} \mathbf{s} \mathbf{T} \mathbf{x}_S \cdot \mathbf{n} = \mathbf{H} \cdot \mathbf{n} \), where \( \mathbf{n} \) is the normal to the wall, is usually imposed (with \( \mathbf{v}_S \cdot \mathbf{n} \) being determined from the requirement that the total mass flux vanish). However, it can be shown that in principle this
condition leads to a violation of the law of increase of entropy (cf. (397) and preceding discussion), and that the boundary conditions (even in the case of a heat current through the wall) should be \( n \cdot v_n = 0 \) and \( n \cdot v_s = 0 \). (In practice, the boundary condition \( \varepsilon_s T v_n \cdot n = h \cdot n \) is an excellent approximation; the internal convection mode of heat transfer in helium II in conjunction with the boundary conditions \( n \cdot v_n = 0 \), \( n \cdot v_s = 0 \) leads to a very thin boundary layer through which \( v_n \cdot n \) rapidly changes from zero to \( h \cdot n / \varepsilon_s T \) (cf. III-B-3; also II-A-1 and III-C-4).) Since the equations for \( v_n \), \( v_s \) now contain second spatial derivatives, one also needs conditions on the tangential components of \( v_n \), \( v_s \). The condition on \( v_n \) is usually taken to be \((v_n - U)_{\text{tangential}} = 0\) (\( U \) is the wall velocity). For the quantity \((v_s - U)_{\text{tangential}}\), Lin has proposed the condition

\[
(n \cdot \varepsilon^{(s)})_{\text{tangential}} = \beta |v_s - U|^2 (v_s - U)_{\text{tangential}}
\]  

(398)

where \( \beta \) is independent of \((v_s - U)\) but may depend on the thermodynamic variables (cf. III-C-3 and III-C-4 for a discussion of this boundary condition). (From the general requirement (397), one may deduce the condition \( n \cdot \varepsilon^{(s)} (v_s - U) \geq 0 \); thus in the event of superfluid slip, the general form of the condition (398)--i.e., a relation between \( n \cdot \varepsilon^{(s)} \)\_tangential and \((v_s - U)\)\_tangential--is already determined by the law of increase of entropy. The rate of energy dissipation at the interface associated with the superfluid slip is \( \beta |v_s - U|^4 \) per unit area.)
3. Reversible flows for \( \lambda \neq 0 \)

When the internal energy depends on the superfluid vorticity, the simplest hydrodynamic equations are obtained from the general equations by taking \( \tau^{(m)}_s, \tau^{(u)}_s, F \) and \( \Phi_s \) all zero. We will limit the discussion of this section to that case.

From equation (383) for the superfluid momentum, we see that there is an additional stress \( \frac{\lambda}{\omega} \omega \omega \) acting on the supercomponent. From equation (385), it is evident that the superfluid vorticity no longer moves with the superfluid, but moves with a velocity \( \nu_s = v_s + \frac{1}{\epsilon_s} \text{curl} \lambda \nu \). (In the present case, \( \epsilon_s = \frac{F}{\epsilon_s} + \frac{1}{\epsilon_s} \text{div} \tau^{(s)}_s \equiv 0 \).) Equation (388) for the scalar vorticity \( \omega = \text{curl} \nu_s \) also illustrates this; the term \( \omega \nu : \text{grad} \nu \) on the right-hand side of (388) represents the production of vorticity by stretching of the vortex lines. The extra terms in the energy flux (cf. (394)) are

\[
\mathcal{Q}' = - \left\{ \frac{\lambda}{\omega} \omega \omega \right\} \cdot \nu_s + \lambda \omega (\nu_s - v_s). \tag{399}
\]

We may interpret the first term in (399) as the rate of working of the extra stresses, provided we assume that the stresses do work on the vorticity velocity \( \nu_s \) rather than \( v_s \); the second term is a correction to the convection term \( \epsilon_s \epsilon_s v_s \), expressing the fact that the vorticity moves with velocity \( \nu_s \) rather than \( v_s \). Alternatively, we may write (399) as

\[
\mathcal{Q}' = - \left\{ \frac{\lambda}{\omega} \omega \omega \right\} \cdot v_s + \lambda \omega (\nu_s - v_s) \text{transverse}. \tag{400}
\]
where

\[(v_\perp - v_3)_{\text{transverse}} = \gamma \times \left\{ (v_\perp - v_3) \times \gamma \right\}; \quad (401)\]

thus we may also regard the vorticity as convected with the velocity \(v_3 + (v_\perp - v_3)_{\text{transverse}}\) rather than \(v_\perp\), and the extra stresses as working on \(v_3\) rather than \(v_\perp\).

The boundary conditions at a wall moving with velocity \(U\) may be taken as \((v_3 - U) \cdot \mathbf{r} = 0\) and \((v_\perp - U) \cdot \mathbf{r} = 0\). Since the equation for \(v_3\) now contains second space derivatives, further boundary conditions are needed. Although it is not obvious what further boundary conditions to impose, the general criterion (397) yields in the present case (of no dissipation)

\[\lambda \frac{\omega \cdot (v_\perp - U)_{\text{transverse}} = 0}{(402)}\]

where

\[(v_\perp - U)_{\text{transverse}} = \gamma \times \left\{ (v_\perp - U) \times \gamma \right\}.\]

Since two further boundary conditions are needed, the simplest choice (satisfying (402)) is

\[(v_\perp - U)_{\text{transverse}} = 0. \quad (403)\]

If one regards the vortex lines as moving with velocity \(v_\perp\), the condition (403) is equivalent to the statement that the point of attachment of a given line to the boundary remains fixed relative to the boundary.
4. Dissipative flows for $\lambda \neq 0$

In the general equations, there will be dissipative terms $\tau^{(n)}$, $\mathcal{E}$, and $F$. (It has not been found possible to include a dissipative stress tensor $\tau^{(m)}$ in the supercomponent in the case $\lambda \neq 0$; the reasons for this will be discussed below. For now we assume that $\tau^{(m)} = 0$.) Again, we make the plausible assumption that the contributions to the dissipation function from the volume mutual friction $F$ and the transport terms $\mathcal{E}$, $\tau^{(n)}$ are independently positive definite.

The quantities $\tau^{(n)}$ and $\mathcal{E}$ are to be determined from the condition (cf. (393))

$$\tau^{(n)} : \nabla \nu - \mathcal{E} \cdot \nabla \Omega \geq 0.$$  

In the simplest case, $\mathcal{E} = -\frac{K}{T} \nabla \Omega$ (K being the thermal conductivity) and $\tau^{(n)}$ is expressed in terms of the normal fluid rate of strain tensor through a shear viscosity and a bulk viscosity. As in the discussion of section 2, we may consider the more general case in which the viscosity tensor and the thermal conductivity tensor depend on the direction of the vector $\omega$. (Because of the distinctive role of the quantity $\omega = \text{curl } \nu$, one could also consider the possibility that the viscosity and conductivity tensor depend on the direction of $\omega$ as well.)

The mutual friction $F$ must be determined so that $F \cdot (\nu - \nu) \geq 0$; again, we can say little more about $F$ until we know what vectors $F$ may depend on. Certainly $F$ will depend on $\nu - \nu$; it
also seems reasonable to suppose that \( \mathbf{F} \) depends on \( \omega = \text{curl} \mathbf{v}_s \) and \( \mathbf{w} = \mathbf{v}_n - \mathbf{v}_s \) as well. The most general \( \mathbf{F} \) depending on these three vectors and satisfying \( \mathbf{F} \cdot (\mathbf{v}_n - \mathbf{v}_s) \geq 0 \) is given by

\[
\mathbf{F} = C_0 (\mathbf{v}_n - \mathbf{v}_s) + (\mathbf{v}_n - \mathbf{v}_s) \times \left\{ C_1 \mathbf{w} + C_2 \omega \times \mathbf{w} + C_3 \omega \times (\mathbf{v}_n - \mathbf{v}_s) \right\} + C_4 \omega \times (\omega \times \mathbf{w}) + C_5 \omega \times (\omega \times [\mathbf{v}_n - \mathbf{v}_s]) \right\}.
\]

\( C_0 \) must be positive, while the other 5 scalar coefficients are arbitrary. (In general, the \( C_i \)'s will depend on the scalar invariants of \( \mathbf{v}_n - \mathbf{v}_s \), \( \mathbf{w} \) and \( \omega \), as well as the thermodynamic variables.) The expression (404) includes--as special cases--the mutual friction given by Bekarevich and Khalatnikov [3] (cf. IV-B-1) and the mutual friction given by Hall [14]. The mutual friction leads to an extra term in the energy flux (cf. (394)), namely

\[
\lambda \omega \left\{ \frac{\mathbf{u} \times \mathbf{F}}{\omega \rho_s^2} \right\}.
\]

From the equation (388) for the vorticity \( \omega = |\text{curl} \mathbf{v}_s| \), we see that the mutual friction affects the transport of vorticity and thus leads to an additional term in the energy flux. In fact, it is convenient to define a modified vorticity velocity \( \tilde{\mathbf{v}}_s \) by

\[
\tilde{\mathbf{v}}_s = \mathbf{v}_s + \frac{\mathbf{v} \times \mathbf{F}}{\omega \rho_s^2}.
\]

Then the energy flux (394) may be written as

\[
\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}^{(m)} \mathbf{v}_n - \frac{A}{\omega} \omega \omega \cdot \tilde{\mathbf{v}}_s + \lambda \omega (\tilde{\mathbf{v}}_s - \mathbf{v}_s) + T \mathbf{J}_s.
\]
For the boundary conditions to be satisfied at a solid boundary, we have, as before, \( \mathbf{n} \cdot (\mathbf{v}_n - \mathbf{U}) \) and \( \mathbf{v}_n - \mathbf{U} = 0 \). Since the equation for \( \mathbf{v}_n \) contains second spatial derivatives, two further conditions are needed. The general criterion (397) yields the condition

\[
\lambda \omega \mathbf{n} \cdot (\mathbf{v}_l - \mathbf{U})_{\text{transverse}} \leq 0. \tag{407}
\]

One possible boundary condition is

\[
(\mathbf{v}_l - \mathbf{U})_{\text{transverse}} = 0; \tag{408}
\]

more generally, we may allow a slip provided (407) is satisfied. The form of the boundary condition in case of a slip depends on what vectors \((\mathbf{v}_l - \mathbf{U})_{\text{transverse}}\) may depend on. Thus if we assume that \((\mathbf{v}_l - \mathbf{U})_{\text{transverse}}\) (at the wall) may depend on \( \mathbf{v} \) and \( \mathbf{n} \), the most general boundary condition satisfying (407) is

\[
(\mathbf{v}_l - \mathbf{U})_{\text{transverse}} = \mathcal{S}_1 \mathbf{v} \times (\mathbf{v} \times \mathbf{n}) + \mathcal{S}_2 \mathbf{n} \times \mathbf{v} \tag{409}
\]

where \( \mathcal{S}_1 \geq 0 \), and \( \mathcal{S}_1, \mathcal{S}_2 \) may depend on \((\mathbf{v} \cdot \mathbf{n})^2\) as well as scalar thermodynamic variables. One may obtain more general slip boundary conditions by allowing the quantity \((\mathbf{v}_l - \mathbf{U})_{\text{transverse}}\) (evaluated at the wall) to depend on \( \mathbf{v} \), as well as \( \mathbf{v} \) and \( \mathbf{n} \).

Finally, we discuss briefly the case when \( \mathbf{v}^{(s)} \neq 0 \). In this case, it is possible to develop a set of hydrodynamic equations similar to Lin's equations for dissipative processes; however, from (393) we see that the relevant rate of strain tensors are those formed from \( \mathbf{v}_n \) and \( \mathbf{v}_l \), rather than \( \mathbf{v}_n \) and
\( \nu \). The difficulty occurs with the boundary conditions; from the general criterion (397), we must have (taking \( \nu - U = 0 \), \( n \cdot (\nu - U) = 0 \))

\[
-\Omega \cdot \frac{\tau_x^{(s)}}{\omega} (\nu_t - \nu) + \tau_x \cdot \lambda \omega (\nu_t - \nu)_{\text{transverse}} + \Omega \cdot \lambda \omega \left( \frac{\nu \times \hat{f}}{\omega} \right) = 0 \tag{410}
\]

in the present case, however, the quantity \( \frac{\nu \times \hat{f}}{\omega} + \frac{1}{\nu_t} \) contains fourth derivatives of \( \nu \) (since \( \nu_t \) already contains second derivatives of \( \nu \)). Because of the condition (410), the boundary conditions must necessarily involve the term \( \lambda \omega \left( \frac{\nu \times \hat{f}}{\omega} \right) \), and thus must involve the fourth derivative of \( \nu \). Thus although the equations may be consistently developed in this case, the difficulty with the boundary conditions seems to preclude the possibility of simultaneously having \( \tau_x^{(s)} \neq 0 \), \( \lambda \neq 0 \).
D. Further Work

Although there are a number of points connected with the present work which deserve further investigation, there are three in particular which we wish to mention briefly here.

The first of these is the difficulty with the variational principle in Lin's one-fluid model (cf. II-B-2; the variational principle entails a restriction on the quantity \( \nabla \times \mathbf{E} \), or, in terms of the two-fluid model, on \( \nabla \times (\mathbf{v}_2 - \mathbf{v}_3) \)). It would be highly desirable to resolve this difficulty so that a unique set of equations could be obtained from the one-fluid model. As discussed in Chapter II (II-B-2; II-C), it is possible that the difficulty stems from the fact that the Hamilton's principle of mechanics was used to obtain the equations of motion, whereas we know that even for reversible flows of helium II energy transfers of an essentially thermal nature may take place.

The second point is concerned with the problem of rotation of helium II. As discussed in Chapter IV (IV-B-3), it is possible that it is an oversimplification to allow the internal energy of helium II to depend directly on the vorticity \( \omega = \nabla \times \mathbf{v}_2 \); one might expect that there will be an internal parameter describing the microscopic rotational state of helium II and that for steady (or slowly varying) flows this internal parameter is approximately equal to \( \omega = \nabla \times \mathbf{v}_2 \). Such a theory has been developed for an ordinary fluid (cf. e.g. [4, p.304]), and it would be of great interest to see whether it is capable of explaining some the peculiar experimental results
for helium II in rotation.

The third point is also concerned with the rotation of helium II, and is simply that it might be of some interest to develop a statistical theory to describe flows containing a large number of vortex filaments (cf. IV-C for some discussion of this).
CONCLUSION

One of the central issues at present in the theory of liquid helium II is the question of superfluid rotation. This is a question which cannot be resolved by continuum principles; rather, one must take a definite view point on this matter in order to develop a hydrodynamic theory. Indeed, the principal point on which the theories discussed in the preceding chapters differ is in their treatment of the superfluid rotation. Ultimately, the choice among the various theories must be made on the basis of comparison with experiment. However, it is not entirely a straightforward matter to obtain a definite set of hydrodynamic equations on the basis of qualitative assumptions about the nature of the superfluid rotation; because of this, it is sometimes difficult to gauge the relevance to the problem of superfluid rotation of particular experimental evidence. A principal aim of the present work has been to attempt to clarify the steps leading from the fundamental assumptions to the hydrodynamic equations in the various theories, both by critically examining the various derivations of the hydrodynamic equations and, in some cases, by offering alternative derivations.
APPENDIX--APPLICATIONS OF LIN'S THEORY OF DISSIPATIVE PROCESSES

We give here an analysis of some simple, specific flow problems on the basis of Lin's theory of dissipative processes (as discussed in section III-C). These examples serve to illustrate some of the features of Lin's theory; also, the comparison of theoretical results with experiment leads to some information about the viscosity coefficients and the coefficient $\beta$ in the nonlinear boundary condition. We use the approximate form of the hydrodynamic equations appropriate for moderate velocities and small temperature variations throughout the flow. These equations are

\[
\frac{\partial \mathbf{v}_n}{\partial t} + \mathbf{v}_n \cdot \nabla \mathbf{v}_n = -\frac{1}{\rho} \nabla p - (\frac{1-x}{x}) S \nabla T - (\frac{1-x}{2}) \nabla w^2 + \frac{\mu_{nn}}{\rho} \nabla^2 \mathbf{v}_n + \frac{\mu_{ns}}{\rho} \nabla^2 \mathbf{v}_s \quad (A-1)
\]

\[
\frac{\partial \mathbf{v}_s}{\partial t} + \mathbf{v}_s \cdot \nabla \mathbf{v}_s = -\frac{1}{\rho} \nabla p + S \nabla T + x \nabla \frac{1}{2} w^2 + \frac{\mu_{sn}}{\rho} \nabla^2 \mathbf{v}_n + \frac{\mu_{ss}}{\rho} \nabla^2 \mathbf{v}_s \quad (A-2)
\]

and

\[
\text{div } \mathbf{v}_n = 0, \quad \text{div } \mathbf{v}_s = 0. \quad (A-3)
\]

In this approximation, $\rho$, $x$, $S$, $\mu_{nn}$, $\mu_{ns}$, $\mu_{sn}$, $\mu_{ss}$ are all taken as constant throughout the flow, and the pressure $p$ and temperature $T$ are then "hydrodynamic" variables. In the problems discussed here, it will not be necessary to consider the case of a heat flux through a boundary wall; thus the boundary conditions (271) may be written as
\[ v_n = U_{wall}, \]

\[ n \cdot (v_s - U_{wall}) = 0, \]

\[ \{ \tau_{ij}^{(s)} n_i - \beta |v_s - U_{wall}|^2 (v_s - U_{wall})_j \}_{\text{tangential}} = 0, \] (A-4)

where

\[ \tau_{ij}^{(s)} = 2 \mu_{sn} \varepsilon_{ij} + 2 \mu_{ss} \varepsilon_{ij}, \]

and \( n \) is the unit normal pointing into the fluid. In section A we consider simple channel flows, in section B we consider the steady flow between rotating cylinders and in section C we consider the Andronikashvili experiment. In section D, some preliminary results on the numerical values of the dissipative coefficients are discussed.
A. Simple Channel Flows

We consider in some detail here the flow induced in an infinitely long channel by constant temperature and pressure gradients along the channel. The height of the channel is 2h (cf. Fig. 3), and it is assumed that the depth is great enough so that the flow may be taken as two-dimensional. Thus we assume that the pressure and temperature are given by $p = -\bar{z} + p_0$, $T = -\bar{T}z + T_0$, where $\bar{p}, \bar{T}$ are the constant gradients, and we look for solutions of the form $v_n = v_n(y) e_z$ and $v_s = v_s(y) e_z$.

From the equations (A-1), (A-2), it follows that $v_n(y), v_s(y)$ are quadratic functions of $y$. The boundary conditions to be satisfied in this case are

$$v_n(\pm h) = 0,$$

and

$$\{ \mu_{sn} \frac{dv_n}{dy} + \mu_{ss} \frac{dv_s}{dy} \} = \bar{T} \beta \{ v_s(\pm h) \}^3.$$

The velocity profiles which satisfy these boundary conditions and the equations are given by

$$v_n(y) = C_n (h^2 - y^2) \quad (A-5)$$

and

$$v_s(y) = C_s (h^2 - y^2) + D_s, \quad (A-6)$$

where

$$C_n = \frac{1}{2 \Delta \mu} \left\{ \bar{p} \left[ x \mu_{ss} - (1-x) \mu_{ns} \right] + \bar{T} \left[ \mu_{ns} + \mu_{ss} \right] \right\}, \quad (A-7)$$
Figure 3
\[ C_s = \frac{1}{2 \Delta \mu} \left\{ \bar{p} \left[ (1 - x) \mu_{nn} - x \mu_{nn} \right] - \varepsilon_s (1 - x) \bar{T} \left[ \mu_{nn} + \mu_{nn} \right] \right\}, \quad (A-8) \]

\[ D_s = \left\{ \frac{h (1 - x) (\bar{p} - \varepsilon_s \bar{T})}{\beta} \right\}^{1/3}, \quad (A-9) \]

where

\[ \Delta \mu = \mu_{nn} \mu_{ss} - \mu_{nn} \mu_{nn}. \quad (A-10) \]

(We should note that, strictly speaking, the temperature must also contain a \( y \)-dependent term; from the equations, one may show that \( T = T_0 - \bar{T} + T'(y) \), where \( T'(y) = -\frac{x}{\varepsilon} \frac{1}{2} \bar{w}^2(y) \). However, even for velocities of the order of \( 10 \, \text{cm sec}^{-1} \), the total variation of \( T \) across the channel is only of the order \( 10^{-5} \, \text{K} \). In most cases of practical interest, this term will be unimportant, and we shall ignore it from here on). The mean velocities are given by

\[ \bar{v}_n = \frac{2}{3} h^2 C_n, \quad (A-11) \]

and

\[ \bar{v}_s = \frac{2}{3} h^2 C_s + D_s, \quad (A-12) \]

the heat current is

\[ W = \varepsilon_s T \bar{v}_n = \frac{2}{3} h^2 \varepsilon_s T C_n, \quad (A-13) \]

and the mass flux is
It is clear from (A-6) and (A-5) that the nonlinear boundary condition allows a slip of the superfluid at the boundary; the magnitude of the slip velocity is determined by the "thermomechanical head" $\bar{p} - \rho \bar{s} \bar{T}$.

It is well-verified experimentally that the heat current in channel flows is proportional to the pressure gradient. It is clear from (A-7) and (A-13) that the present theory predicts this provided

$$\mu_{ss} + \mu_{sn} = 0,$$  \hspace{1cm} (A-15)

and in the remainder of the Appendix, we will assume (A-15) to hold. In this case, $C_n$ is given by

$$C_n = \frac{\bar{p} \mu_{ss}}{2 \Delta \mu} = \frac{\bar{p}}{2(\mu_{nn} + \mu_{sn})},$$  \hspace{1cm} (A-16)

while $C_s$ and $D_s$ are still given by (A-8) and (A-9). Thus the pressure gradient alone is the driving force for the normal fluid, whereas both $\bar{p}$ and $\bar{T}$ are driving forces for the superfluid. For the combination $v_s + \frac{\mu_{sn}}{\mu_{ss}} v_n$, however, we have

$$v_s + \frac{\mu_{sn}}{\mu_{ss}} v_n = \frac{\beta}{\hbar \mu_{ss}} \frac{1}{2} D_s \left( h^2 - y^2 \right) + D_s,$$  \hspace{1cm} (A-17)

so that the "thermomechanical head" $\bar{p} - \rho \bar{s} \bar{T}$ alone is the driving force for $v_s + \frac{\mu_{sn}}{\mu_{ss}} v_n$. 

\[ M = \frac{4}{3} \hbar^3 \rho \left( x \, C_n + (1-x) \, C_s \right) + 2 \hbar \rho (1-x) D_s. \]  \hspace{1cm} (A-14)
It is of interest to consider the special case when there is no net mass flux (internal convection); then the (nonlinear) relation between the pressure gradient $\bar{p}$ and the temperature gradient $\bar{T}$ is given (implicitly) by

$$\frac{2}{3} \times h^2 C_n + \frac{2}{3} (1-x) h^2 C_s + (1-x) D_s = 0. \quad (A-18)$$

For $h \to 0$, one may show from (A-18) that

$$\bar{p} = \rho s \bar{T} + O(h^5) \quad (A-19)$$

(or, more explicitly,

$$\bar{p} = \rho s \bar{T} \left\{ 1 - \frac{\beta (\rho s \bar{T})^2}{27 \mu_s^3 (1-x)^4} \left( \frac{x \mu_{ss} - (1-x) \mu_{ssn}}{\mu_{nn} + \mu_{ssn}} \right)^3 h^5 \right\},$$

so that the London equation is recovered in the limit of $h \to 0$.

For larger values of $h$, the expression for the heat current as a function of the temperature gradient may be written as

$$W = \frac{h^2 \rho s \bar{T}}{3(\mu_{nn} + \mu_{ssn})} \rho s \bar{T} \left\{ 1 - \frac{\lambda}{3} \frac{\Delta^3}{h^3} \right\}, \quad (A-20)$$

where

$$\lambda = \frac{x}{1-x} - \frac{\mu_{ssn}}{\mu_{ss}} \left\{ \frac{x}{1-x} + (1-x) \left( \frac{\mu_{nn} + \mu_{ssn}}{\mu_{ss}} - \frac{\mu_{ssn}}{\mu_{ss}} \right) \right\} \quad (A-21)$$

where $\Delta^3$, a dimensionless measure of the temperature gradient, is given by

$$\Delta^3 = \rho s \bar{T} \Delta$$
where

$$\frac{1}{\rho} = \left( \frac{\nu}{1-x} - \frac{\mu_s}{\mu_s} \right) \left( \frac{\nu}{1-x} + \frac{(1-x)(\mu_{nn} + \mu_{sn})}{\mu_{ss}} - \frac{\mu_{sn}}{\mu_{ss}} \right)^{\nu/2} \beta^{1/2} \frac{h^{5/2}}{3 \sqrt{1-x} (\mu_{nn} + \mu_{sn})^{3/2}}$$  \hspace{1cm} (A-22)

and where \( \hat{s} \) is the (unique) root of

$$\frac{1}{3} \hat{s}^3 + \hat{s} = \hat{t}. \hspace{1cm} (A-23)$$

As a function of \( \hat{t} \), the quantity \( \frac{\lambda \hat{s}^3}{3 \hat{t}} \) increases monotonically from zero to \( \lambda \) as \( \hat{t} \) increases from zero to infinity. (From the restrictions (258) on the viscosity coefficients, one may show that \( \lambda < 1 \), so that \( \mathcal{W}/\tau \) is always positive.) If we introduce

$$W_0 = \frac{h^2 \rho \mathcal{S} \mathcal{T} \mathcal{R}}{3 (\mu_{nn} + \mu_{sn})}, \hspace{1cm} (A-24)$$

then

$$W = W_0 \left\{ \hat{t} - \frac{\lambda}{3} \hat{s}^3 \right\} \hspace{1cm} (A-25)$$

or

$$W = W_0 \left\{ (1-\lambda) \hat{t} + \lambda \hat{s} \right\},$$

and

$$W = W_0 \hat{t} \left\{ 1 + O\left( \hat{t}^3 \right) \right\}, \hspace{0.5cm} \hat{t} \to 0 \hspace{1cm} (A-26)$$

$$W = W_0 (1-\lambda) \hat{t} \left\{ 1 + O\left( \hat{t}^{-2/3} \right) \right\}, \hspace{0.5cm} \hat{t} \to \infty. \hspace{1cm} (A-27)$$

Another case of interest is the isothermal flow in a channel under a pressure gradient. In this case, the relation between the mean flow velocity and the pressure gradient is
\[ \bar{v} = x \bar{v}_n + (1-x) \bar{v}_s = \frac{h^2 \bar{p}}{3 (\mu_{nn} + \mu_{sn})} \left\{ x + (1-x)^2 \left( \frac{\mu_{nn} + \mu_{sn}}{\mu_{ss}} \right) - (1-x) \frac{\mu_{sn}}{\mu_{ss}} \right\} + (1-x) \frac{(1-x)^{1/2}}{\beta} \bar{p}^{1/2}. \tag{A-28} \]

The calculations for the flow in an infinitely long circular pipe of radius \(a\) are entirely similar, and we only quote some of the results here. The profiles and mean velocities are (we are assuming \(\mu_{nn} + \mu_{ss} = 0\) still)

\[ v_n(r) = C_n (a^2 - r^2), \tag{A-29} \]
\[ v_s(r) = C_s (a^2 - r^2) + D_s, \tag{A-30} \]
\[ C_n = \frac{\bar{p}}{4 (\mu_{nn} + \mu_{sn})}, \tag{A-31} \]
\[ C_s = \frac{1}{4 \mu_{ss} (\mu_{nn} + \mu_{sn})} \left\{ \bar{p} \left[ (1-x) \mu_{nn} - x \mu_{ss} \right] - (1-x) \rho s \bar{T} (\mu_{nn} + \mu_{sn}) \right\}, \tag{A-32} \]
\[ D_s = \left\{ \frac{a (1-x) (\bar{p} - \rho s \bar{T})}{2 \beta} \right\}^{1/2} \tag{A-33} \]
\[ \bar{v}_n = \frac{1}{2} C_n a^2, \quad \bar{v}_s = \frac{1}{2} C_s a^2 + D_s. \tag{A-34} \]

For internal convection, the relation between the heat current and the temperature gradient is

\[ W = \frac{a^2 \rho s \bar{T}}{8 (\mu_{nn} + \mu_{sn})} \left[ 1 - \frac{\lambda}{4} \frac{\bar{s}^3}{\bar{T}} \right], \tag{A-35} \]
where

\[ \lambda = \frac{x}{1-x} \frac{\mu_{sn}}{\mu_{ss}} \left\{ \frac{X}{1-X} \frac{\mu_{sn}}{\mu_{ss}} + \frac{(1-x)(\mu_{nn} + \mu_{sn})}{\mu_{ss}} \right\}^{2} \]  \hspace{1cm} (A-36)

where

\[ \hat{T} = \frac{\mathcal{E} \mathcal{S} \mathcal{T}}{\mathcal{S}} \]

with

\[ \mathcal{S} = \frac{8(1-x)(\mu_{nn} + \mu_{sn})^{3/2}}{(\frac{X}{1-X} - \frac{\mu_{sn}}{\mu_{ss}})(\frac{X}{1-X} - \frac{\mu_{sn}}{\mu_{ss}} + \frac{(1-x)(\mu_{nn} + \mu_{sn})}{\mu_{ss}})^{1/2}} \alpha^{5/2} \beta^{1/2} \]  \hspace{1cm} (A-37)

and where \( \mathcal{S} \) is the root of

\[ \frac{1}{4} \mathcal{S}^{3} + \mathcal{S} = \hat{T} \]  \hspace{1cm} (A-38)

For isothermal flow under pressure gradient, the mean velocity is given by

\[ \bar{v} = x \bar{v}_{n} + (1-x) \bar{v}_{s} = \frac{a^{2} \bar{p}}{8(\mu_{nn} + \mu_{sn})} \left\{ x + (1-x)^{2} \frac{\mu_{nn} + \mu_{sn}}{\mu_{ss}} - (1-x) \frac{\mu_{sn}}{\mu_{ss}} \right\} \]

\[ + (1-x) \left\{ \frac{(1-x)a^{2/3}}{2 \beta} \right\} \bar{p}^{1/3} \]  \hspace{1cm} (A-39)
B. Flow Between Rotating Cylinders

We now consider the steady flow between two concentric cylinders, the inner cylinder being of radius $a$ and having angular velocity $\Omega_a$, and the outer cylinder being of radius $b$ and having angular velocity $\Omega_b$ (cf. Fig. 4). We look for solutions of the form $v_s = v_s(r) \theta, v_n = v_n(r) \hat{e}_\theta$; it follows from the equations that

$$v_n(r) = A_n r - \frac{B_n}{r}, \quad (A-40)$$

and

$$v_s(r) = A_s r - \frac{B_s}{r}. \quad (A-41)$$

$v_n(r)$ must satisfy

$$v_n(a) = a \Omega_a, \quad v_n(b) = b \Omega_b, \quad (A-42)$$

so that $A_n, B_n$ are given by

$$A_n = \Omega_a + \frac{\gamma^2 \Omega \omega_{rel}}{\gamma^2 - 1}, \quad (A-43)$$

$$B_n = \frac{\gamma^2 a^2 \Omega \omega_{rel}}{\gamma^2 - 1}, \quad (A-44)$$

where

$$\gamma = \frac{b}{a}, \quad \Omega \omega_{rel} = \Omega_b - \Omega_a. \quad (A-45)$$

The boundary conditions for $v_s(r)$ are

$$\left\{ \mu_{sn} \frac{d}{dr} \left( \frac{v_n}{r} \right) + \mu_{ss} \frac{d}{dr} \left( \frac{v_s}{r} \right) \right\}_{r=a}^{r=a} = \beta \left\{ v_s(a) - a \Omega_a \right\}^3, \quad (A-46)$$
\[ \{ \mu_{sn} r \frac{d}{dr} \left( \frac{v_n}{r} \right) + \mu_{ss} r \frac{d}{dr} \left( \frac{v_s}{r} \right) \}_r \text{rs} = - \beta \left\{ v_s(b) - b \Omega b \right\}^3 \]  

(A-47)

so that \( A_s, B_s \) satisfy

\[ \frac{2}{d^2} \left[ \mu_{sn} B_n + \mu_{ss} B_s \right] = \beta \left\{ A_s a - \frac{B_s}{a} - a \Omega a \right\}^3 \]  

(A-48)

and

\[ \frac{2}{b^2} \left[ \mu_{sn} B_n + \mu_{ss} B_s \right] = - \beta \left\{ A_s b - \frac{B_s}{b} - b \Omega b \right\}^3. \]  

(A-49)

The solutions for \( A_s, B_s \) may be written as

\[ A_s = A_n - \frac{\gamma^2 \left( \gamma^{\frac{1}{2}} + 1 \right)}{(\gamma^2 - 1)(\gamma^2 + \gamma^{\frac{1}{2}})} \left( \frac{\mu_{sn} + \mu_{ss}}{\mu_{ss}} \right) \Omega_{rel} (1-q)^2 \]  

(A-50)

\[ B_s = B_n \left\{ 1 - \left( \frac{\mu_{sn} + \mu_{ss}}{\mu_{ss}} \right) (1-q) \right\} \]  

(A-51)

where \( q \) is the (unique) root of

\[ 2q = \Omega^2 (1-q)^3, \]  

(A-52)

where

\[ \Omega^2 = \frac{\beta \left( \frac{\mu_{sn} + \mu_{ss}}{\mu_{ss}} \right)^2 a^3 \gamma^4 \Omega_{rel} (\gamma^2 - 1)}{\mu_{ss} (\gamma^2 + \gamma^{\frac{1}{2}})^3}. \]  

(A-53)

The parameter \( q \) varies from 0 to 1 as \( \Omega^2 \) varies from 0 to \( \infty \). For \( \Omega_{rel} = 0 \), \( q \) vanishes, and in this case the normal fluid and the superfluid move together in a rigid body rotation. In general, there is a slip of the superfluid at the boundary;
thus
\[ a \Omega_a - v_s(a) = \left( \frac{\mu_{sn} + \mu_{ss}}{\mu_{ss}} \right) \frac{a^2 \gamma^2 \Omega_{rel} (1 - q)}{\gamma^2 + \gamma^3} , \]  
(A-54)

and
\[ b \Omega_b - v_s(b) = \left( \frac{\mu_{sn} + \mu_{ss}}{\mu_{ss}} \right) \frac{a \gamma \Omega_{rel} (1 - q)}{\gamma^{5/3} + 1} , \]  
(A-55)

so that, also,
\[ \{ a \Omega_a - v_s(a) \} = - \gamma^{2/3} \{ b \Omega_b - v_s(b) \}. \]  
(A-56)

Thus the velocity of the superfluid relative to the wall is always of opposite sign for the two boundaries, and the magnitude of the slip is greater (by a factor of $\gamma^{2/3}$) at the inner wall.

A particular case of interest is the flow in a rotating cylinder viscometer; in this case we have $\Omega_a = 0$, and we wish to calculate the torque (per unit length) on the inner cylinder as a function of $\Omega = \Omega_b$. The calculation is easily made from the above results, and the torque is given by
\[ T = 4\pi \gamma^2 a^2 \Omega \left( \frac{\mu_{nn} \mu_{ss} - \mu_{ns} \mu_{sn}}{\mu_{ss}} \right) \left[ 1 + q \left( \frac{\mu_{ss} + \mu_{sn}}{\mu_{nn} \mu_{ss} - \mu_{ns} \mu_{sn}} \right) \right] . \]  
(A-57)

From the channel flow experiments, we have concluded that
\[ \mu_{ns} + \mu_{ss} = 0 ; \]  
(A-58)

it then follows that
\[ T = 4\pi \gamma^2 a^2 \Omega \left( \mu_{nn} + \mu_{sn} \right) . \]  
(A-59)
Thus the torque is a linear function of $\Omega$ (in agreement with the experimental results $[13]$), and the viscosity measured by such an experiment is the combination $\mu_{nn} + \mu_{sn}$.

If we also impose the Onsager relation

$$\mu_{ns} = \mu_{sn} \gamma$$

(A-60)

so that

$$\mu_{sn} = -\mu_{ss},$$

then it can be shown that in all cases of steady flow between rotating cylinders, the superfluid component moves with the normal fluid. By way of contrast, we may note that in the case $\mu_{sn} = 0$, the superfluid and normal fluid in general do not move together.

For convenience of reference, we note here that the angular momentum (per unit length) of a flow between rotating cylinders is given by

$$L = \frac{\pi \rho a^4}{2} \left\{ (\gamma^2 - 1) \Omega_n + \gamma^2 \Omega_{rel} - (1 - \lambda) \frac{\mu_{sn} + \mu_{ss}}{\mu_{ss}} \right\} \cdot (1 - \eta \lambda) \Omega_{rel} \frac{\gamma^2 (\gamma^2 - 1)}{\gamma^2 + \gamma^3}.$$  

(A-61)
C. Andronikashvili Experiment

Only the simplest aspects of the problem will be treated here—no consideration is given to edge effects or nonlinear convective terms, the oscillating surfaces are taken as infinite in extent, and the nonlinear boundary condition is treated only in the low-speed approximation (for which it is a linear boundary condition).

In the problem to be considered, Lin's hydrodynamic equations may be reduced to the system

\[
\begin{align*}
\rho_n \frac{\partial v_n}{\partial t} &= \mu_{nn} \frac{\partial^2 v_n}{\partial z^2} + \mu_{ns} \frac{\partial^2 v_s}{\partial z^2}, \\
\text{and} \\
\rho_s \frac{\partial v_s}{\partial t} &= \mu_{sn} \frac{\partial^2 v_n}{\partial z^2} + \mu_{ss} \frac{\partial^2 v_s}{\partial z^2},
\end{align*}
\tag{A-62}
\]

where \( v_n \), \( v_s \) are velocity components parallel to the oscillating surfaces and \( z \) is the space coordinate perpendicular to the surfaces. Solutions of (A-62) that are periodic in time may be found in the form \( e^{i\omega t} e^{pz} \). It is convenient to introduce the kinematic viscosities

\[
\begin{align*}
\nu_{nn} &= \frac{\mu_{nn}}{\rho_n}, & \nu_{ss} &= \frac{\mu_{ss}}{\rho_s}, \\
\nu_{ns} &= \frac{\mu_{ns}}{\sqrt{\rho_n \rho_s}}, \text{ and } & \nu_{sn} &= \frac{\mu_{sn}}{\sqrt{\rho_n \rho_s}};
\end{align*}
\tag{A-63}
\]

it is also convenient to introduce

\[
\Delta \nu = \nu_{nn} \nu_{ss} - \nu_{ns} \nu_{sn},
\tag{A-64}
\]
\[ V = \sqrt{\left( V_{nn} + V_{ss} \right)^2 - 4 \Delta V} \quad \text{(A-65)} \]

(We assume here that \( V \) is real; in particular, this is true if \( \mu_{ns} \mu_{sn} \geq 0 \), which includes the cases \( \mu_{sn} = \mu_{ns} \quad \mu_{sn} = 0 \).)

Then the solutions of (A-62) are the form

\[
\begin{pmatrix} \frac{p_n}{p_s} \\ \frac{v_n}{v_s} \end{pmatrix} = e^{i \omega t} \left\{ \left( ae^{P_+ Z} + be^{-P_+ Z} \right) \left( \frac{2 V_{ns}}{V_{ss} - V_{nn} - V} \right) \right. \\
+ \left. \left( ce^{P_- Z} + de^{-P_- Z} \right) \left( \frac{2 V_{ns}}{V_{ss} - V_{nn} + V} \right) \right\}, \quad \text{(A-66)}
\]

where

\[
P_+ = \frac{1}{2} \left( 1 + i \right) \sqrt{\frac{\omega}{\Delta V}} \sqrt{V_{nn} + V_{ss} + V}, \quad \text{(A-67)}
\]

and

\[
P_- = -\frac{1}{2} \left( 1 + i \right) \sqrt{\frac{\omega}{\Delta V}} \sqrt{V_{nn} + V_{ss} - V}.
\]

Thus in the present case there are two viscous penetration depths given by

\[
\delta_{\pm} = \frac{1}{\Re (P_\pm)} = 2 \frac{\Delta V}{\sqrt{\omega (V_{nn} + V_{ss} \pm V)}}, \quad \text{(A-68)}
\]

The nonlinear boundary condition to be satisfied at a wall is

\[
\mu_{sn} \frac{\partial v_n}{\partial Z} + \mu_{ss} \frac{\partial v_s}{\partial Z} = \beta \left( v_s - U_{\text{wall}} \right)^3, \quad \text{(A-69)}
\]
For sufficiently small wall velocities, this condition is, approximately,

\[ \mu_{sn} \frac{\partial \nu_n}{\partial z} + \mu_{ss} \frac{\partial \nu_s}{\partial z} = 0, \quad \text{(low-speed)} \quad (A-70) \]

whereas for sufficiently high wall velocities, the condition is

\[ \nu_s = U_{wall}, \quad \text{(high-speed)}. \quad (A-71) \]

We now consider the Andronikashvili experiment, taking the idealized situation of the flow between two oscillating parallel infinite discs. The discs are located at \( z = \pm h \), and the boundary velocity is taken to be

\[ \nu_{\text{disc}} = \xi_\theta Ar \omega \cos \omega t, \]

where \( r \) is the polar radius measured from the axis of rotation.

We look for solutions of the form

\[ \nu_n (r, z, t) = \xi_\theta r \nu_n (z, t) \quad (A-72) \]

\[ \nu_s (r, z, t) = \xi_\theta r \nu_s (z, t). \]

Then \( \nu_n, \nu_s \) must satisfy the equations (A-62). We consider here only the low-speed limit of the boundary conditions, so that

\[ \nu_n (\pm h, t) = A \omega \cos \omega t = \Re A \omega e^{i \omega t} \]
\[ \mu_{sn} \frac{\partial \nu_n}{\partial z} (\pm h, t) + \mu_{ss} \frac{\partial \nu_s}{\partial z} (\pm h, t) = 0. \]  \hfill (A-73)

The solutions for \( \nu_n, \nu_s \) may be written in the form

\[
\begin{pmatrix} \sqrt{\frac{F_n}{F_s}} \nu_n \\ \sqrt{\frac{F_n}{F_s}} \nu_s \end{pmatrix} = \frac{\sqrt{\frac{F_n}{F_s}}} {2 \nu_{ns}} \begin{pmatrix} \cosh P_z Z & 2 \nu_{ns} \frac{\nu_{ss} - \nu_{nn} - \nu}{\nu_{ss} - \nu_{nn} - \nu} + \frac{1 - 2 \nu_{ns} \nu_{ns} + \nu_{ss} \nu_{ss} - \nu_{nn} - \nu}{2 \nu_{ns} \nu_{ns} + \nu_{ss} (\nu_{ss} - \nu_{nn} - \nu)} \end{pmatrix} \left[ \frac{P \tanh P_h}{P \tanh P_h} \right]^{-1} \begin{pmatrix} \cosh P_z Z & 2 \nu_{ns} \frac{\nu_{ss} - \nu_{nn} + \nu}{\nu_{ss} - \nu_{nn} + \nu} - \frac{\cosh P_z Z \nu_{ss} - \nu_{nn} - \nu}{\cosh P_z Z \nu_{ss} - \nu_{nn} - \nu} \end{pmatrix}, \hfill (A-74)\]

The shears are given by

\[
\frac{\partial \nu_n}{\partial z} \bigg|_h = 2 \omega e^{i \omega t} \nu_{ss} P \tanh P_h \tanh P \bigg[ \frac{2 \nu_{nn} \nu_{ns} + \nu_{ss} (\nu_{ss} - \nu_{nn} + \nu)}{2 \nu_{nn} \nu_{ns} + \nu_{ss} (\nu_{ss} - \nu_{nn} - \nu)} \bigg] \frac{P \tanh P_h}{P \tanh P_h}, \hfill (A-75)\]

and

\[
\frac{\partial \nu_s}{\partial z} \bigg|_h = - \frac{\mu_{sn}}{\mu_{ss}} \frac{\partial \nu_n}{\partial z} \bigg|_h. \]

A particular case of interest is when the penetration depths are large compared with the disc spacing \( h \); then, approximately,

\[
\frac{\partial \nu_n}{\partial z} \sim \frac{A \omega^2 e^{i \omega t}}{\Delta_\nu} \left( \frac{h \nu_{ss}}{1 \nu \nu_{ss}} \right), \quad \text{for} \; |P | h \ll 1 \hfill (A-76)\]

and the order of magnitude condition for the validity of the low-speed approximation to the boundary condition may be written as

\[
\omega \ll \frac{\rho h}{\beta R^2 A^2}, \hfill (A-77)\]

where \( R \) is the radius of the oscillating disc.
The angular momentum of the fluid contained between the two discs of radius $R$, separation $2h$ is given by

$$L = \frac{\pi R^4 \rho_n}{\omega} \frac{\Delta \eta}{\eta_{ss}} \frac{\partial v_n}{\partial z} \bigg|_h$$  \hspace{1cm} (A-78)

in the limit $|\rho_n| h \ll 1$, we have

$$L = \pi R^4 \rho_n A \omega h e^{i\omega t}$$  \hspace{1cm} (A-79)

If the space between the discs were filled with a solid of density $\rho_n$, it is easy to show that the angular momentum would also be given by (A-79). Thus in the limit of large penetration depths and low speeds, only the normal fluid is carried by the discs— in agreement with experiment—and this conclusion is independent of any restrictive assumptions on the viscosity coefficients (such as $\mu_{sn} = 0$, or $\mu_{ns} = \mu_{nn}$).
D. Comparison with Experiment

Ultimately, one would hope to obtain values for the quantities $\mu_{nn}$, $\mu_{ns}$, $\mu_{sn}$, $\mu_{ss}$ and $\beta$ as functions of temperature from the analysis of various experiments. In the present section, we give some preliminary results along these lines. Although the experimental evidence for the relation $\mu_{ns} + \mu_{ss} = 0$ is convincing, it would be very desirable to obtain some experimental evidence relevant to deciding whether $\mu_{ns} = \mu_{sn}$ or $\mu_{sn} = 0$. (Of course, it is possible that neither of $\mu_{ns} = \mu_{sn}$ or $\mu_{sn} = 0$ holds; since there are, however, arguments in favor of both of these relations, we will not in the present section consider the more general case.) The preliminary results given here tend to favor the case $\mu_{ns} = \mu_{sn}$.

We consider first experiments designed to measure the viscosity of helium II—the damping of an oscillating disc and the rotating cylinder viscometer. According to Lin's theory, the rotating cylinder viscometer measures the combination $\mu_{nn} + \mu_{sn}$ (cf. section B of this Appendix). For an ordinary fluid, the damping of an oscillating disc serves as a measure of the product $\rho \mu$. In the usual interpretation of the experimental results for helium II, it is assumed that the relevant density is $\rho_n$ and the relevant viscosity is $\mu_n$. The oscillating disc experiment may be analyzed on the basis of Lin's theory, and the quantity $\mu_n$ expressed in terms of the parameters of Lin's theory. Table 1 shows which quantities in Lin's theory play the role of the effective viscosity. Thus it is clear that if $\mu_{sn} = 0$, the effective viscosities
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Note: $x = \frac{e_n}{\mu_{nn}}, \quad r = \frac{\mu_{ss}}{\mu_{nn}}$

**Table 1. Effective Normal Fluid Viscosity From Lin's Theory**
should be the same; however, the measured results are not the same (see, e.g., Atkins [2], p. 105), the discrepancy being particularly great at low temperatures, where the viscosity from the disc experiments is significantly larger. In the case $\mu_{sn} = \mu_{ns}$, Lin's theory predicts that the effective viscosities will be different; in fact, for low temperatures ($x \to 0$), the effective viscosity in the disc experiment tends to $\mu_{nn}$, whereas the effective viscosity in the rotating cylinder viscometer is $\mu_{nn} - \mu_{ss}$. For high temperatures ($x \to 1$), both effective viscosities tend to $\mu_{nn} - \mu_{ss}$, and the experimental results show that the discrepancy between the measured viscosities is indeed smallest near the $\lambda$-point. The experimental results then seem to indicate the choice $\mu_{ns} = \mu_{sn}$. However, there are several reasons for regarding this as only a tentative conclusion: (i) in order to calculate the effective viscosity from the disc experiments, one must know the normal fluid density $\rho_n$, and it is difficult to accurately determine the quantity at the lower temperatures for which the discrepancy between the two effective viscosities is greatest, (ii) an accurate measurement of the damping of the disc is more difficult at lower temperatures, and (iii) the theory (in the case $\mu_{ns} = \mu_{sn}$) predicts that the disc effective viscosity should always be greater than the rotating cylinder viscosity, whereas the experimentally determined values of the disc viscosity seem to be somewhat smaller than the rotating cylinder viscosity for temperatures above 1.8 K. In the case $\mu_{sn} = \mu_{ns}$, the experimental results ([2], p. 105) indicate that (i) at lower temperatures ($T \approx 1.4$ K) $\mu_{nn} \approx 20 \cdot 10^{-6}$ poise and
\[ \mu_{ss} \sim 5 \cdot 10^{-6} \text{ poise} \] (ii) at higher temperatures (near the \( \lambda \) - point) \( \mu_{nn} - \mu_{ss} \sim 20 \cdot 10^{-6} \text{ poise} \) and (iii) at intermediate temperatures (\( T \sim 1.8^\circ \text{K} \)) \( \mu_{nn} - \mu_{ss} \sim 12 \cdot 10^{-6} \text{ poise} \).

Another possible experimental test to choose between and \( \mu_{ns} = \mu_{sn} \) is suggested by the equation (A-61) for the angular momentum of the flow between rotating cylinders. In particular, for the choice \( \Omega_a = -\gamma \Omega_b \), the angular momenta for the two cases are

\[ \mu_{ns} = \mu_{sn} : \quad L = 0 \quad (A-80) \]
and
\[ \mu_{sn} = 0 : \quad L = -\pi \frac{a^4 (1-x)(1-\gamma) \Omega_b (\gamma^{-1})^2 (\gamma^{\gamma-1})}{\gamma^2 + \gamma^{\gamma+1}}. \quad (A-81) \]

Thus it should be possible to obtain some evidence relevant to this question provided the angular momentum can be measured with an accuracy sufficient to discern (A-80) from (A-81). Although Reppy and Lane [52] have made measurements of the angular momentum of helium II contained in a single rotating cylinder, it would presumably be much more difficult in the case of fluid contained between two rotating cylinders. The situation is further complicated by the fact that if \( \Omega_b \) is too large, the parameter \( \gamma \) (cf. (A-52), (A-53)) will be close to unity; on the other hand, \( \Omega_b \) must be large enough so that the measurement can choose between (A-80) and (A-81).

As discussed in part C of this Appendix, the Andronikashvili experiment does not choose between \( \mu_{ns} = \mu_{sn} \) and \( \mu_{sn} = 0 \), since the theory predicts that (for sufficiently large
penetration depths and sufficiently small amplitudes of oscillation) only the normal component moves with the discs. However, it is possible to obtain some order of magnitude estimates of the boundary constant $\beta$ from some oscillating disc-pile experiments of Hollis-Hallett [17]. He found that above a certain critical amplitude, the period of oscillation was observed to increase, indicating that the discs were carrying a fraction of the total liquid greater than $\frac{P_o}{P}$. This is agreement with Lin's theory, in which--because of the nonlinear boundary condition--increasing entrainment of the superfluid component occurs with increasing amplitude of oscillation. From Hollis-Halleyt's data, some (very crude) estimates of $\beta$ were made. In the temperature range 1.7°K to 2.14°K, values of $\beta$ in the range $4 \times 10^{-3} - 5 \times 10^{-2}$ (cgs) were found; although the estimates were far too crude to determine the temperature dependence of $\beta$, there was some indication of a maximum in the vicinity of 1.8°K. For temperatures above 1.5°K, all of the $\beta$ values were in the range $2 \times 10^{-2} - 5 \times 10^{-2}$ (cgs). These estimates are at best only order of magnitude.

Further evidence for the choice $\mu_{s\nu} = \mu_{ns}$ may be obtained from the results of some pipe flow experiments by Staas, Tacconi and van Alphen [34]. They have studied both laminar and turbulent flows in a pipe in the parabolic approximation when the temperature and pressure gradients are related by $\frac{\partial T}{\partial s} = \frac{\partial P}{\partial s}$. The onset of turbulence was determined experimentally by noting when the heat current ceased to be proportional to the pressure gradient. They found that over a wide range of
temperatures (and for pipes of three different diameters) the onset of turbulence corresponded to a value of 1200 for the Reynolds' number based on the pipe diameter, the mean normal fluid velocity, the normal fluid viscosity $\mu_n$ and the total fluid density. The analysis of this experiment on the basis of Lin's theory yields the following results: (i) for $\mu_{sn} = 0$, only the normal component moves, so that the density of the moving fluid is $\rho_n$; (ii) for $\mu_{ns} = \mu_{sn}$, the normal and supercomponents move together, so that the density of the moving fluid is $\rho$. Since the experimentally determined critical Reynolds' number based on the total density $\rho$ is constant over a wide range of temperatures, this would seem to be further evidence for the choice $\mu_{ns} = \mu_{sn}$.

In principle, the experimental results for various channel and pipe flows afford the best opportunity for obtaining accurate numerical values for the viscosity coefficients and the boundary constant $\beta$. In practice, however, one must be sure that the flows analyzed are laminar. The work of Staas et al. described above has given a criterion for turbulence in the case that $\bar{p} = \bar{\rho} \bar{v}^2$. However, in their experiments, the normal and supercomponents move together (in the case $\mu_{ns} = \mu_{sn}$), so that the critical Reynolds' number—which was based on the mean fluid velocity—could equally well have been based on the mean mass velocity or the mean superfluid velocity. In more general pipe flows, there may be two critical Reynolds' numbers; even so, the criterion furnished by Staas et al. should be a valuable guide in the interpretation of these more
general flows. Of course, even for laminar flows one cannot discount entirely the possibility of a volume mutual friction force and an associated critical velocity. In spite of these difficulties, the pipe and channel flows still seem to be the most likely source of accurate values of the viscosities and the boundary constant $\beta$.
CITED REFERENCES


The author was born May 5, 1936 at Elizabethton, Tennessee. The author attended Purdue University (Sept., 1954 -- June, 1958) and was awarded the B.S. degree in Engineering Sciences in June, 1958. The author has been in attendance at M.I.T. from Sept., 1958 until the present, with the exception of one summer (1962) with the International Business Machines Corporation at Yorktown Heights, New York. While at M.I.T., the author was a research assistant (Sept. 15, 1958 -- Sept. 15, 1961; Sept 15, 1962 -- present) and also held an NSF Cooperative Graduate Fellowship (Sept. 15, 1961 -- June 15, 1962).