Figure 1. The **Jumbo Jet** at Coney Island, New York. An example of a horizontal loop. ROBERT API

A Primer on Roller Coaster Dynamics

Part II – You Can Bank on Them

by Prof. Alfred Clark, Jr.

Sometimes Down Is Sideways

Tim Onosko reports an interesting anecdote John Allen told him in an interview [1, p. 56]. One of the old-time coaster designers once asked Allen why he banked his track on the turns. Allen's answer was that he did it to keep the passengers in the car! In this part of our Primer we'll see just how engineers like Allen calculate the bank angle required for a given speed and track curvature.

Let's begin with the basic seat force equation developed in Part I. We have

$$S + m(g - a) = 0, \qquad (1)$$

where m is the mass of the passenger, g is the acceleration of gravity, a is the acceleration of the coaster car, and S is the seat force on the passenger. As discussed in Part I, this may be viewed as an equilibrium of the passenger under the action of three forces: the seat force S, the gravity force mg, and the inertial force (−ma). In Part II, it is more convenient to write equation (1) as

$$S = m(a - g). \qquad (2)$$

With it, we can calculate the seat force S if we know a and g. This seat force doesn't depend on the orientation of the coaster car — only on gravity and the acceleration of the car. However, that's a bit misleading. It makes quite a difference to the passenger whether the required seat force S comes from the bottom of the seat or the side of the car.

Either way equation (2) is satisfied, but the two sensations are entirely different. So, to describe the passenger experience completely, we must deal with the orientation of the coaster car — a natural extension of Part I, where we dealt only with the car position.

Let's start with a special case that's both simple and interesting: the near-horizontal loop typified by the *Jumbo Jet* at Coney Island (Figure 1) or by coasters of the runaway-mine-train type. These loops, not quite horizontal, are helices of very shallow pitch wound around vertical cylinders. The excitement comes from the steep banking of the car. If you have the good seat on the inside of the curve, you hang out over empty space.

Let's analyze this situation for a circular loop which, for simplicity, we assume is horizontal. The coaster car acceleration **a** is horizontal and points toward the center of the loop. It is given by $\mathbf{a} = -(V^2/R)\mathbf{i}$, where V is the speed, R is the radius of curvature, and **i** is a unit vector pointing away from the center of the loop. The seat force equation (2) becomes

$$S = -m[(V^2/R)\mathbf{i} + g]. \qquad (3)$$

Figure 2 shows a diagram of the three forces. Given this seat force **S**, we may bank the track at an angle ψ so that the required seat force is all provided by the bottom of the seat, with no component required from the side of the car. (This is the situation shown in Figure 2.) This does two desirable things. First, it prevents the passenger from being thrown out the side of the car, and second, it gives the hanging-out-over-space configuration that we discussed above. The necessary bank angle ψ is easily shown to be

$$\psi = \tan^{-1}(V^2/Rg). \qquad (4)$$

Of course, the closer ψ is to 90 degrees, the more exciting the ride. It isn't hard to get an exciting ride. If, for example, V = 30 mi/h (modest) and R = 25 ft (typical), then $\psi = 67$ degrees. (A reminder: we must use consistent units in numerical calculations. A convenient set is feet (ft) for lengths, seconds (s) for times, and feet per second (ft/s) for velocities. The value of g in these units is 32.2 ft/s^2. A useful conversion factor is 1 mi/h = 1.47 ft/s.) In an actual coaster where the track is a shallow descending helix, the speed and the bank angle increase as the train progresses downward, so there is a nice buildup to the final angle which, for example, will exceed 80 degrees if V exceeds 50 mi/h. In this 80-degree bank, the passenger feels a seat force from the bottom of the seat, as though gravity were also tilted by 80 degrees. This is a state we can reasonably describe as one in which down is (nearly) sideways.

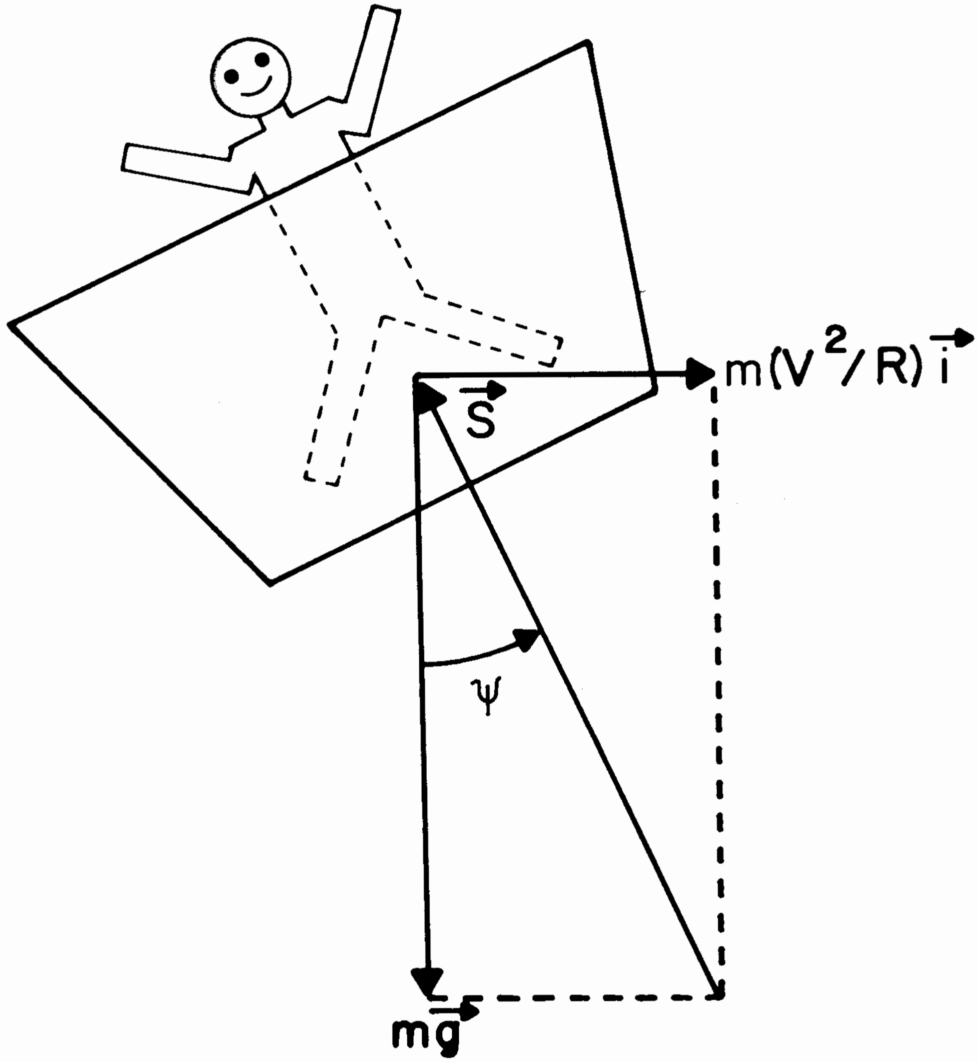


Figure 2. Bank angle ψ on a horizontal loop. The seat for **S** is the negative of the resultant of the gravity force mg and the centrifugal force (mV²/R)**i**. In the situation shown, ψ has been chosen so that the seat force is normal to the seat bottom.

The Inertial Force can be Your Seat Belt

Intensely three-dimensional. That's the only way to describe a coaster like the *Z-Force* (Figure 3). To deal with the car orientation on such a coaster, we need to be more systematic than we have been so far. Figure 4 shows some of the basic geometry for describing the car orientation.

The car can rotate about three axes. One is in the direction of **t**, the tangent vector to the track. Rotation about this axis is called bank. A second axle is defined by the vector **m** perpendicular to the floor of the coaster car. Rotation about **m** is called yaw. The third axis, specified by the vector **b**, is perpendicular to the other two. Rotation about this axis is called pitch. In an airplane, pilots control all three modes of rotation by manipulating the various control surfaces. However, in a coaster car, two of the three angles, corresponding to yaw and pitch, are completely determined by the space curve of the track, because the car always points along the tangent vector **t**. However, the third angle, corresponding to bank, is independent and is not determined by the tangent alone. That means that a complete description of the track geometry calls for a space curve, with a bank angle given at each point of the space curve.

Consider the specification of the bank angle, as shown in Figure 5. At any point on the track, we can construct a plane normal to the track. We call this the transverse plane. Next, we introduce $\mathbf{G}_T$, the projection of the gravity vector in the transverse plane:

$$\mathbf{G}_T = \mathbf{g} - (\mathbf{g}\cdot\mathbf{t})\mathbf{t}. \qquad (5)$$

Then $\mathbf{G}_T$ defines a direction which we call absolute down. A second down, which we call the local down, is the direction normal to the floor of the coaster car. The bank angle ψ is defined as the angle between absolute down and local down.

Can we design a coaster and specify a track curve with attached bank angle so that, in principle, no seat belt or lapbar is needed? The issue here is whether we can orient the car at each point so that the seat force required by equation (2) can be supplied by the bottom or the back of the seat, with no component required from the side of the car or from a lap constraint.

To answer this question, let's recall our simplified force description of Part I, which ignores friction and any other forces parallel to the direction of motion other than gravity. That means the tangential

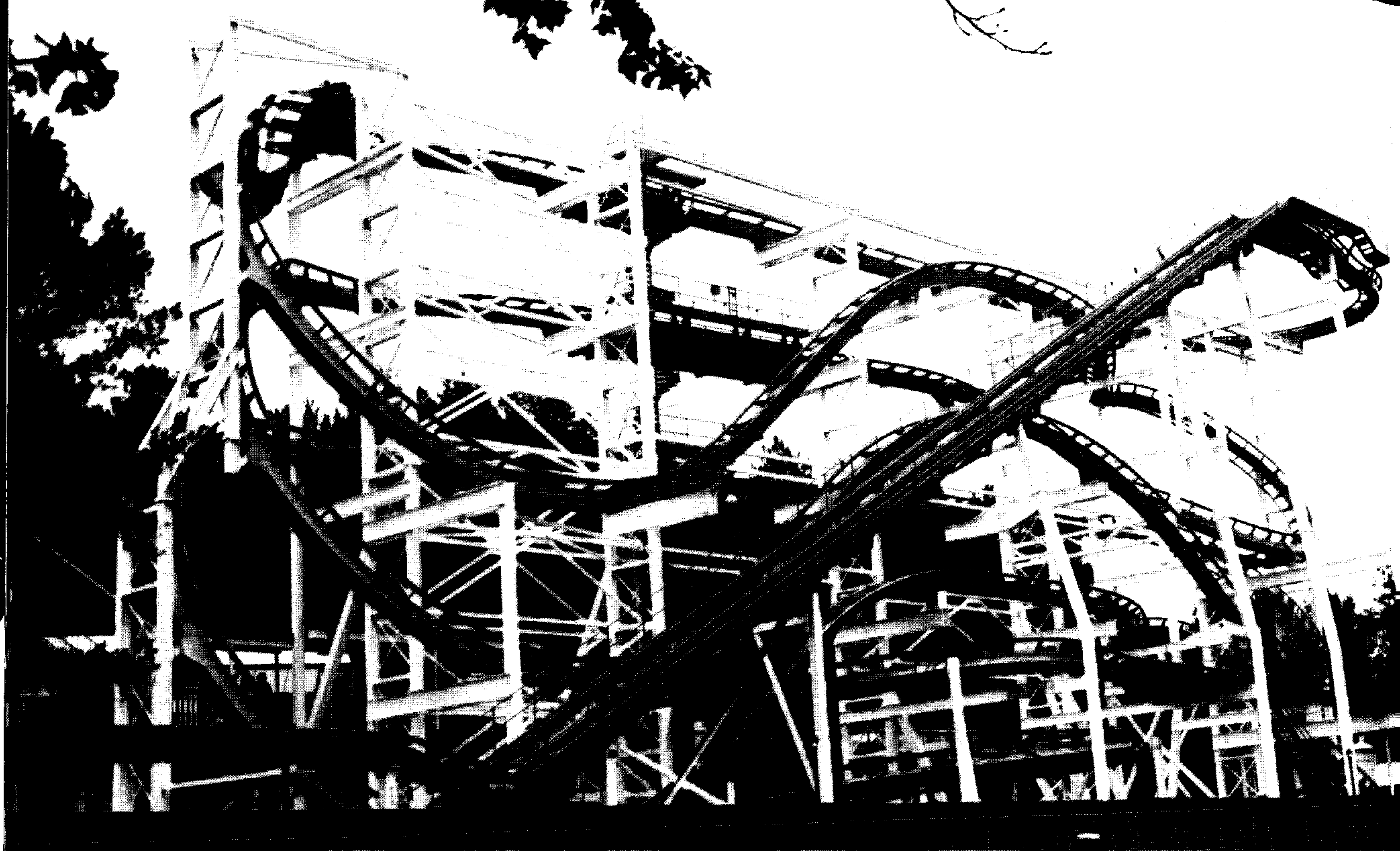


Figure 3. The **Z-Force** at Six Flags Over Georgia, Atlanta. An example of a fully three-dimensional track. SIX FLAGS OVER GEORGIA

acceleration of the coaster car is $a_t = g{\cdot}t$, which is the component of gravity in the tangential direction. The normal acceleration is V^2/R, where V is the speed of the car and R is the radius of curvature. Thus the vector acceleration is

$$\mathbf{a} = (g{\cdot}t)\mathbf{t} + (V^2/R)\mathbf{n}, \quad (6)$$

where, as before, **n** is a unit vector pointing toward the center of curvature of the track. By substituting equation (6) into the seat force equation (2) and using equation (5) for G_T, we get

$$S = m[(V^2/R)n - G_T]. \quad (7)$$

Because **n** and G_T are in the transverse plane, it follows from equation (7) that the seat force S is also in the transverse plane. Then we can choose the bank angle so that the seat force is oriented along local down (as in Figure 2). For such a bank angle, down, as perceived by the passenger, coincides with local down, although the magnitude of the effective gravity will vary as the car speed and track curvature vary. In principle no seat belt or lap bar is necessary. Interestingly enough, such a coaster has been built. The *Zambezi Zinger*, an Intamin-Schwarzkopf *Speedracer* at Worlds of Fun in Kansas City "... allows riders in toboggan-style trains to cruise the computer designed and fabricated track without seat belts or lap bars." [1, p. 45] According to Onosko, the installation of the *Zambezi Zinger* was extremely difficult, requiring great precision in the track layout.

There is an easier way to do it. If the coaster car is suspended like a pendulum, it will continually adjust its bank angle so that the seat force is always along local down. Demonstrating this is a nice exercise for those of you skilled in mechanics. The correctness of the result depends on the mass of the car support member being much less than that of the car, which is true in typical designs. If you are skilled in advanced techniques in dynamics, you might want to go further and tackle the question of whether the car oscillates significantly in a pendulum mode as it is required to change from one bank angle to another by changing speed and track curvature.

Of course, if you'd rather not solve differential equations, you can just ride such a coaster and get an experimental answer to the question. One place to do this is Cedar Point in Sandusky, Ohio, where the *Iron Dragon* resides. Some interesting comments on this coaster by its designer, Ron Toomer, appear in an interview by Paul Ruben [2]. Another suspended coaster is the *Big Bad Wolf* at Busch Gardens in Williamsburg, Virginia. There is an appealing simplicity about these suspended coasters, because of the way in which dynamics automatically produces the desired orientation.

The Curve is Pitched

No, it isn't baseball. It's the fabulous disorienting *Corkscrew* coaster pioneered by Arrow Development (Figure 6). (The first one of these helical marvels appeared at Knott's Berry Farm in 1975 [1, p. 52].) Thanks to gravity, the seat force on a spiraling passenger has an interesting cyclical variation, because the angle between the gravity force and the local down of the passenger is continually changing, and because the speed of the coaster changes as its elevation changes.

Let's see what we can learn about these cyclical forces from an analysis. This calculation is more intricate than the ones we've done so far, although it rests on the same principles. There are three parts to the calculation. We must first specify

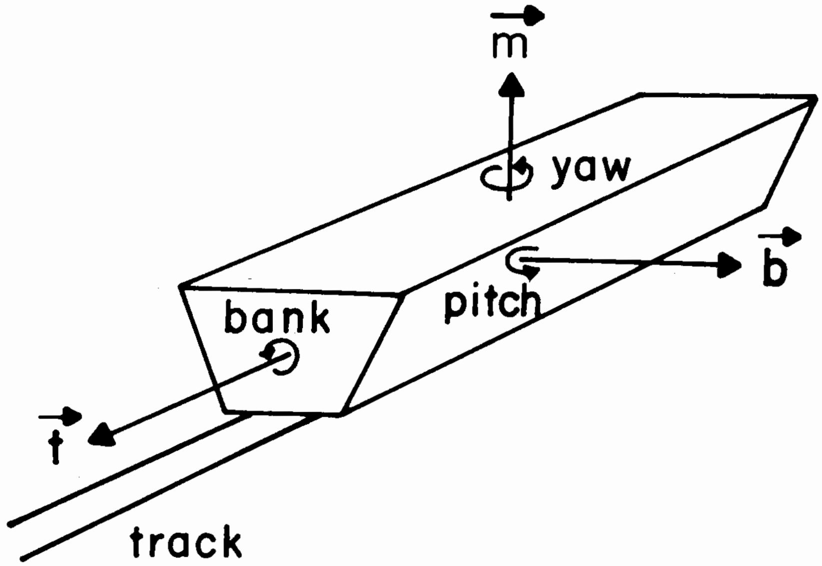


Figure 4. The orientation of a coaster car. Rotation about the tangent vector **t** is bank, rotation about the normal to the car bottom **m** is yaw, and rotation about the vector **b** is pitch.

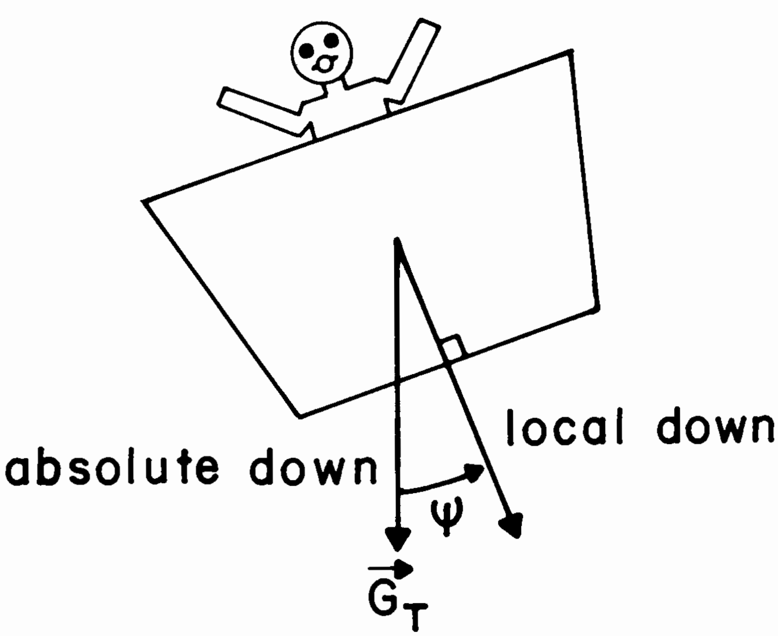


Figure 5. Definition of bank angle. The view is in the plane transverse to the tangent vector **t**. Absolute down is defined to be the direction of $\mathbf{G}_T$, the projection of gravity **g** in the transverse plane. Local down is the direction of the normal to the car floor. The bank angle ψ is the angle between the two.

precisely the geometry of the helical path. Then we must use the seat force equation (7), taking our expressions for the normal **n** and the transverse gravity $\mathbf{G}_T$ from the geometric analysis. Finally, we must use conservation of energy to relate the speed V to the position. Those who prefer not to be dragged through this tripartite analysis can skip on to Figure 8, where the results for the passenger force are shown.

We begin with the geometry (see for example [3] for the necessary information on helices). We choose a coordinate system so that x is horizontal, y is up, and z is horizontal, with positive z being the direction of advance of the coaster. Figure 7 shows a schematic of the geometry. The projection of the helical path onto the x-y plane is a circle of radius a. We let θ be the angle specifying the location of the car on that circle. We assume that the car goes around the circle in the counterclockwise direction (i.e. θ increases with time) and, at the same time, advances in the positive z-direction. As the car makes one complete revolution, it travels a circumferential distance $2\pi a$. At the same time, it will advance in the z-direction. The ratio of the advance per revolution to the circumference is called the pitch p of the helix. The pitch is often specified by giving the pitch angle which is $\tan^{-1}(p)$.

It is useful to introduce a triad of reference vectors which move along with the car. These mutually perpendicular unit vectors are shown in Figure 7. The first is the by now familiar tangent vector t, which points in the direction of the car motion. The second is the normal vector n, which points toward the center of the curvature. The third vector is the binormal e (given by $e = t \times \mathbf{n}$), which is perpendicular to both **t** and **n**.

Now we must be quantitative about the geometry. To reduce the necessary blizzard of equations to a mere flurry, let's summarize the results we need without giving the derivations. (Those of you with some knowledge of vector analysis and space curves can easily derive the results given below). The equation of the helix is given by specifying the position vector r as a function of the angle of revolution θ:

$$\mathbf{r} = a\cos(\theta)\mathbf{i} + a\sin(\theta)\mathbf{j} + pa\,\theta\mathbf{k}. \qquad (8)$$

Here θ is in radians, and **i**, **j**, and **k** are unit vectors in the x, y, and z directions. From this expression for **r**, we can obtain expressions for the tangent vector t, the normal vector **n**, and the binormal vector e:

$$t = -(a/c)\sin(\theta)\mathbf{i} + (a/c)\cos(\theta)\mathbf{j} + (pa/c)\mathbf{k},$$

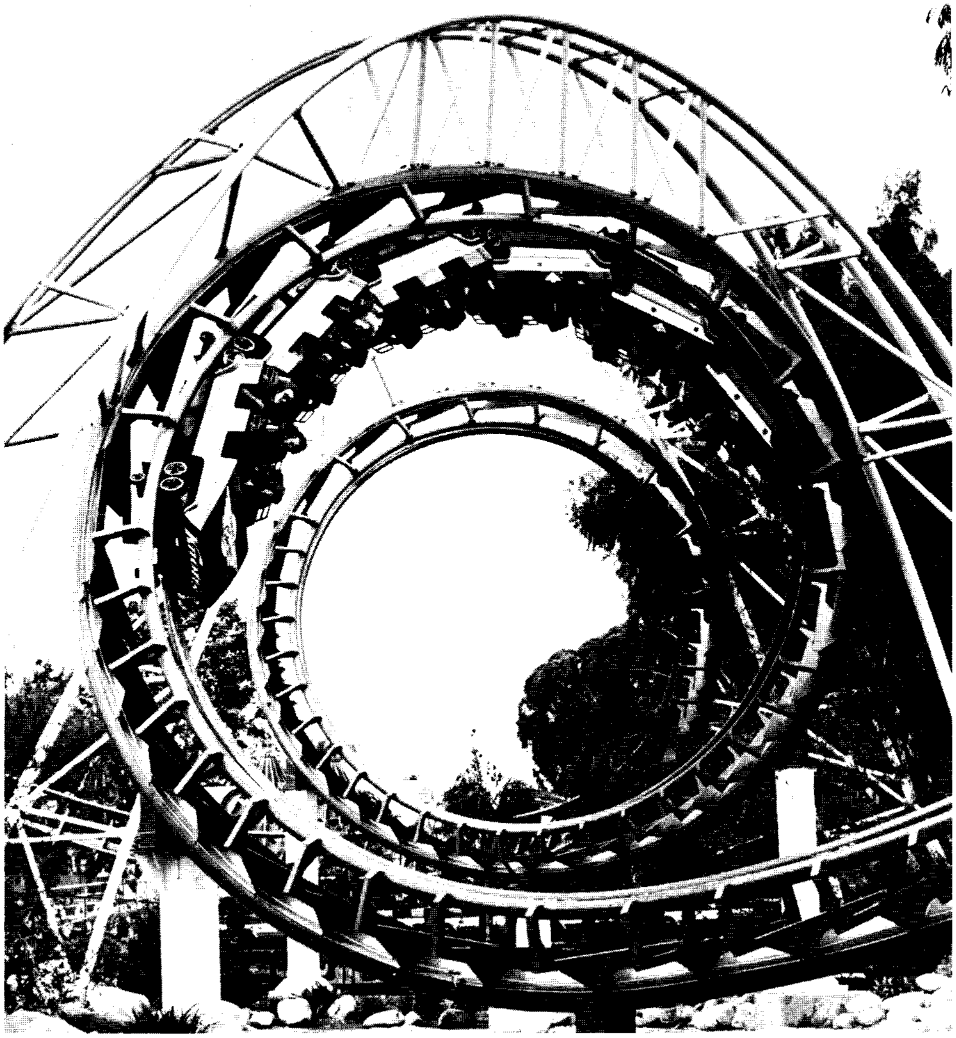
$$\mathbf{n} = -\cos(\theta)\mathbf{i} - \sin(\theta)\mathbf{j}, \qquad (9)$$

and $e = p(a/c)\sin(\theta)\mathbf{i} - p(a/c)\cos(\theta)\mathbf{j} + (a/c)\mathbf{k},$

where $c^2 = a^2(1 + p^2)$.

Finally, we also need the radius of curvature R, given by

$$R = c^2/a. \qquad (10)$$



KNOTT'S BERRY FARM

Figure 6. The first Arrow Development **Corkscrew** coaster at Knott's Berry Farm, Buena Park, California.

Why, you may ask, are all these trig functions and vectors necessary? The situation is a typical one in mechanics. We want to describe events in the natural reference frame of the passenger, so we use the vectors **t**, **n**, and **e**, which are attached to the car. Newton's laws, however, require us to describe events in a fixed reference frame, specified by the vectors **i**, **j**, and **k**. To carry out the analysis, it is necessary to establish the connection between these two frames, and that is exactly what we've just done.

Now we begin the second part of the analysis, the calculation of the seat force. By expressing the unit vector **j** in terms of the vectors **t**, **n**, and **e** and by using equation (5) with $\mathbf{g} = -g\mathbf{j}$, we get for the transverse gravity

$$G_T = g[p(a/c)\cos(\theta)\mathbf{e} + \sin(\theta)\mathbf{n}]. \quad (11)$$

The seat force then follows from equation (7):

$$S = m[V^2/R - g\sin(\theta)]\mathbf{n} - m[gp(a/c)\cos(\theta)]\mathbf{e}. \quad (12)$$

To complete the calculation, we need to find the speed V as a function of angle. We do this by using conservation of energy. For frictionless motion of the car, the sum of the kinetic energy $(mV^2/2)$ and the gravitational potential energy mgy is a constant E. From equation (8) for the car position **r**, we get $y = a\sin(\theta)$. We may evaluate the constant energy E at the low point of the trajectory, where $\theta = 270$ degrees, and $y = -a$. At that point the speed V has a maximum value which we denote by V_M. Then the energy equation is

$$(mV^2/2) + mga\sin(\theta) = E = (mV_m^2/2) - mga,$$

hence $$V^2 = V_M^2 - 2ga[1 + \sin(\theta)]. \quad (13)$$

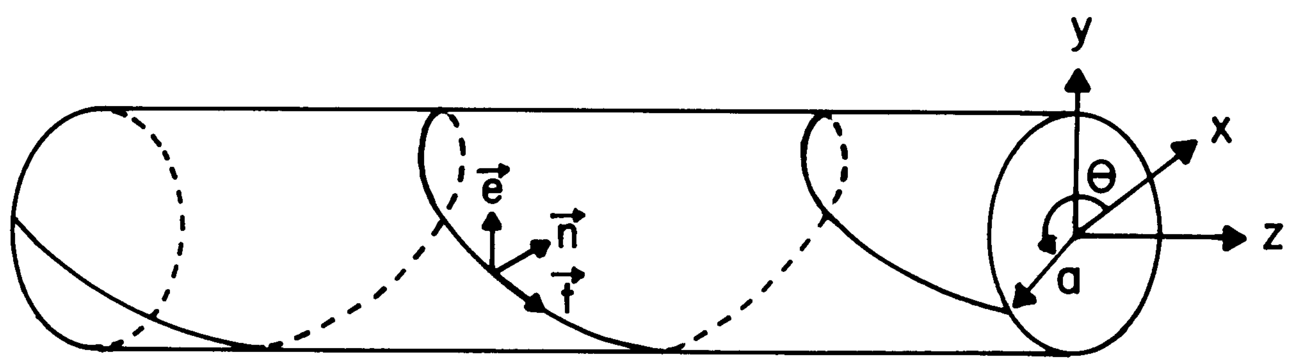


Figure 7. A schematic of the helical geometry of the corkscrew track. The projection of the track on the xy plane is a circle of radius a. The coaster advances in the positive z-direction as it rotates. The angle of rotation in the x-y plane is θ. The local orientation of the car is described by the track tangent vector **t**, the unit normal **n** pointing to the center of curvature, and the binormal **e**.

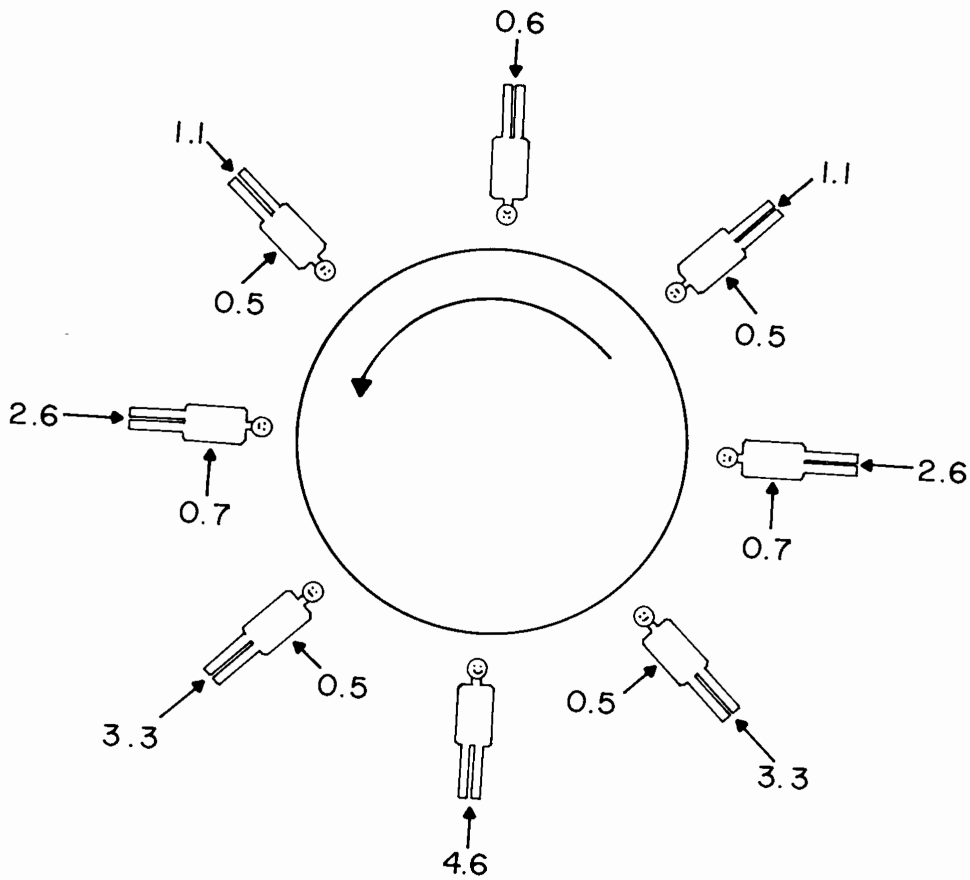


Figure 8. Forces (in gs) on a passenger in a helical loop. The coaster is rotating in the direction of the arrow and advancing out of the paper. The helix pitch angle is 45 degrees, the loop radius is 15 ft, and the maximum speed is 40 mi/h.

By using equations (12) and (13) we get the final formula for the seat force as a function of orientation:

$$S = mg[F_n \mathbf{n} + F_e \mathbf{e}], \qquad (14)$$

where the dimensionless functions F_n and F_e are the number of g's in the **n** and **e** directions, and are given by

$$F_n = (V_M^2/gR) - (2a/R) - (1 + 2a/R)\sin(\theta), \qquad (15)$$

and $$F_e = -p(a/c)\cos(\theta). \qquad (16)$$

If the coaster car is not banked (i.e., if local down is toward the center of the helix), then F_n is the force on the bottom of the passenger from the seat, and F_e is the force on the left side of the passenger (F_e negative means a force on the right side). Let's look at some numbers for a typical case and see in fact whether the coaster needs to be banked. For the parameter values, we take a = 15 ft and a pitch angle of 45 degrees so that p = tan(45) = 1, which gives a radius of curvature of 30 ft. We take the maximum speed to be 40 mi/h. Then the above formulas give

$$F_n = 2.56 - 2\sin(\theta),$$
$$F_e = -0.707\cos(\theta). \qquad (17)$$

Figure 8 shows the cycle of forces on the intrepid passenger. The normal force F_n varies from a low of 0.6 g's at the top to a high of 4.6 g's at the bottom. The sideways component is on the right side of the passenger on the ascending part of the loop, and on the left side during the descent. The maximum side force is 0.7 g's, which is possibly a little large for comfort. By banking the track, we could transfer some of that to the bottom.

No Equations Beyond This Point

Apologies for tossing so many equations into what is supposed to be reading for fun. It could have been worse. The actual design calculations for a coaster, including things we have left out such as friction and multi-car effects, are far more intricate. I hope that what we have done gives you some feeling for how the passenger experience is determined by geometry, and how dynamics is the link between them.

In the upcoming Part III of this Primer, we will extend our dynamical analysis to deal with the interesting and real differences between the first car, the middle car, and the last car. We'll show why the middle car is the tamest, something all experienced coaster riders know. Until then, may the inertial force be with you!

References

[1] **Funland USA**, Tim Onosko, Ballentine Books, New York, 1978.

[2] "Mr. Steel Coaster — Arrow's Ron Toomer," Paul L. Ruben, *RollerCoaster!* Magazine Vol. VIII, Issue 2, pp. 16-17, 1987.

[3] **Differential Geometry**, Dirk J. Struik, 2nd edition, Addison-Wesley, Boston, Massachusetts, 1961.