# A Primer On Roller Coaster Dynamics

## *Part III—Passengers Entrained*

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**Figure 1. *American Eagle* at Great America, Gurnee, Illinois. For a train going down hill with the center of curvature below the track, there are tension forces in the couplings. These forces alter the speeds from the values they would have for uncoupled cars in the same positions. The first car is held back and the last car is pulled forward by the coupling force. Passengers in the first car slide forward, and passengers in the last car are pushed back in their seats.**

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### Is the First Car Worth Fighting For?

The only serious pushing and scuffling I've ever seen in coaster lines has been among those seeking the first seat in the first car. Some coasters even have a special line for that position to help keep the peace.

What is the magic of the first car? Is it really that much better than the other cars? Certainly the first car's awesome view of the drops offers a clear visual and psychological advantage over the other cars, but is there also a dynamical difference? Do passengers in different cars experience a different history of forces? These are the questions we deal with in this part of our primer.

If we have identical coaster cars, not coupled, moving one after another on the same track, all of the passengers have the same sequence of experiences. But when we couple these cars, they exert forces on one another, and these forces alter the speed and acceleration.

Consider the example shown in Figure 1, where we have a train going down a curved hill. Let's look at the effect of the coupling on the first car. If the cars were not coupled, the first car would move ahead of the others, because it's on a steeper downslope. When the cars are coupled, this cannot happen; instead, a force develops in the coupling, holding back the first car and reducing its speed. To develop this qualitative argument into a quantitative statement about speed, we need to extend to coaster trains the speed-height relation that we developed for uncoupled cars in Parts I and II (*RollerCoaster!*, Volume IX Issues 3-4 and Volume X Issue 1). The principle involved is, as before, conservation of energy.

There is a second interesting consequence of the coupling force. For passengers in the first car, the retarding force from the coupling feels just like someone has put on the brakes. As a result of this force, the passenger tends to slide forward. We will study this effect quantitatively by reconsidering the seat force equation, this time in a form appropriate for coaster trains. (As a practical by-product of our analysis, we'll discover the circumstances in which eyeglasses are most likely to fly off!)

In an attempt to maintain some semblance of readability throughout this discussion, I have relegated the more technical details to the Appendix.

## Trading Height for Speed

The power of energy conservation as a tool to solve mechanics problems astonishes me as much today as it did when I first saw it, as a sophomore at Purdue, more than 30 years ago. For many problems, it provides a quick answer completely free of extraneous details. For a coaster train, it leads directly to the speed-height relation that we seek, without requiring us to calculate the forces in the couplings between cars.

The basic principle is that, in the absence of friction, the kinetic energy plus the gravitational potential energy is constant. Using this principle in Part I for a single coaster car led to the relation $V^2 = 2g(H - z)$, where V is the speed of the car, g is gravity, z is the height of the car above ground level, and H is the height of the top of the lift hill above ground level.

For a coaster train, the situation is complicated by the fact that different parts of the train are at different heights, but all parts must move at the same speed because of the coupling. The key concept in analyzing this situation is the center of mass, defined as the average position of all the distributed mass of the train and passengers. For a train on a straight track, the center of mass is, as you might expect, in the middle car. But for a train in the upper part of a tight loop, for example, the center of mass is well below the track. Thus the center of mass is a mathematically defined point that may or may not actually be in the train.

Why do we need this mathematical abstraction? Because it greatly simplifies the application of the principle of conservation of energy. As shown in the Appendix, the relation between speed and height for a coaster train is given by

$$V^2 = 2g(H - z_C) \qquad (1)$$

where V is the speed of the train, $z_C$ is the

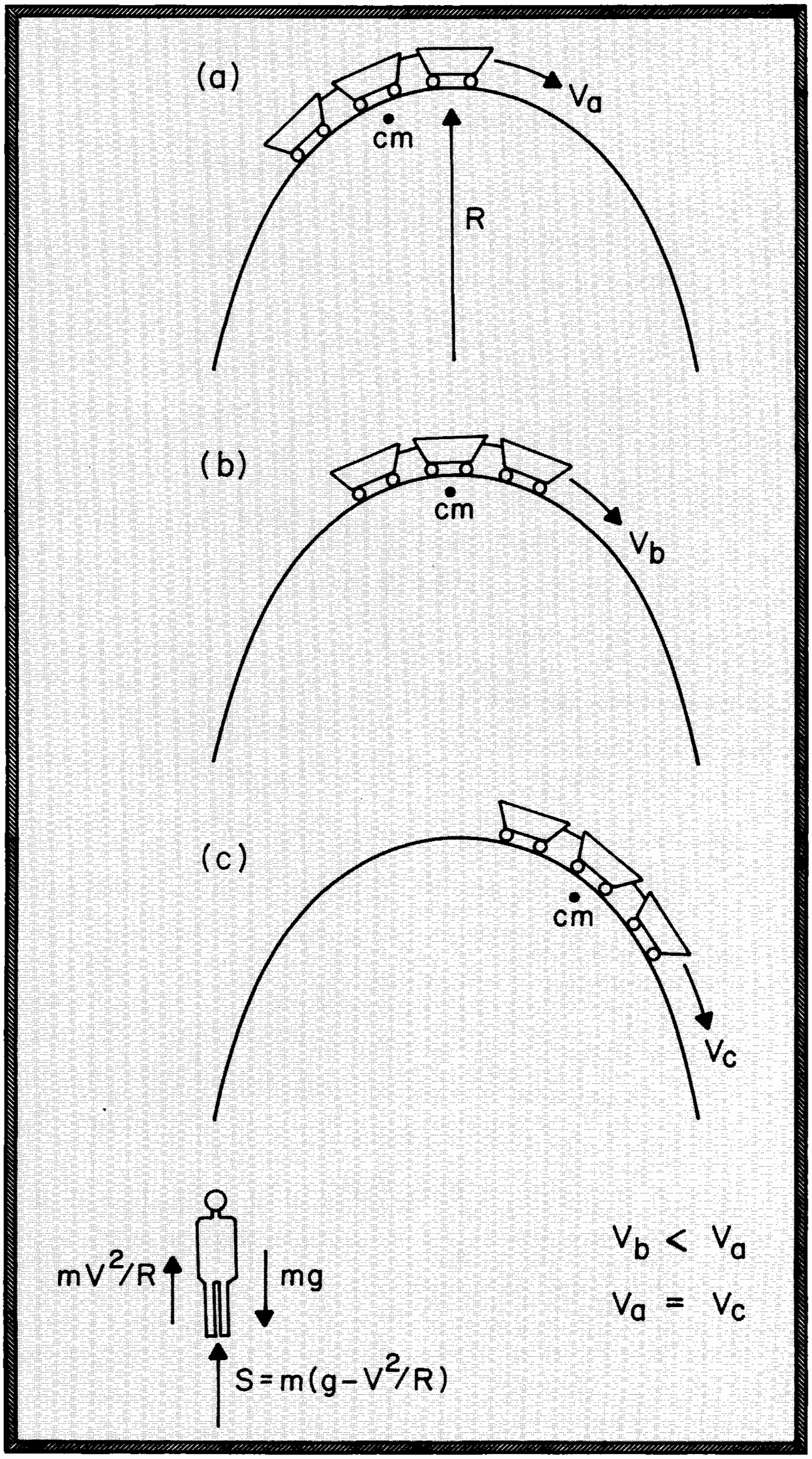



**Figure 2. Coaster train at a symmetric hilltop: (a) first car at the top; (b) middle car at the top; (c) last car at the top. The radius of curvature is R. The position of the center of mass is cm. The cm is lower in cases (a) and (c) than in case (b), so, by conservation of energy, the speed V is greater in (a) and (c) than in (b). The vertical component of the seat force is $S = m(g - V^2/R)$. The upward inertial force $mV^2/R$ tending to lift the passenger out of the seat is least in the middle car.**

height of the center of mass above ground level, and H is the height of the top of the lift hill. This relation has the same simple form as the earlier one for a single coaster car; the only difference is that, for a train, we use the height of the center of mass, rather than the height of any one car. To use equation (1), we need to know how to calculate the position of the center of mass, but this can be done without great difficulty (see the Appendix for more details).

Let's apply this speed-height relation to two interesting situations: hilltops and hill bottoms. We begin with hilltops. As we saw in Part I, the inertial force exerted at a hilltop is opposite to gravity and has magnitude $mV^2/R$, where R is the radius of curvature and m is the mass of the passenger. So, if $mV^2/R$ exceeds our weight mg, we are lifted out of our seats, and only lapbars or belts prevent us from becoming human cannonballs.

Now let's see how this effect varies from car to car in a coaster train. Figure 2 shows a sequence of three positions of a train on a symmetric hill: (a) the first car at the hilltop, (b) the middle car at the hilltop, and (c) the last car at the hilltop. In each case, the position of the center of mass, labeled cm, is shown. The center of mass is highest for case (b), when the middle car is at the top of the hill. Equation (1) tells us the hilltop speed is lowest for the middle car. You already know that from experience—the middle car gives the tamest ride at the top of a hill.

Now let's consider the first and last cars. Here the result may surprise you. If the hill is symmetric about the peak, the center of mass is at the same height when the first car [case (a)] and the last car are at the top [case (c)]. The speed is the same in both cases, and the inertial force $mV^2/R$ lifting us off the seat is the same. Does this mean the rides experienced in the first and last cars are identical? Any experienced coaster rider knows the answer to that question is "no," no matter what equation (1) says.

There are actually two important differences between the first and last cars on a hilltop. The first difference is psychological and results from our perceptions: we remember whether the car we're riding has been slowing down or speeding up. The coaster train slows down throughout the climb, until the middle car reaches the top of the hill. From that point on, it speeds up as the center of mass descends.

Thus, the first car reaches the hilltop in a less exciting sequence (slowing down) than the last car (speeding up). We often describe this by saying the last car is "whipped" over the top of the hill.

The second difference between the first and last car at a hilltop is a true dynamical difference, but we'll put off discussing it until we have the seat-force equation for a coaster train.

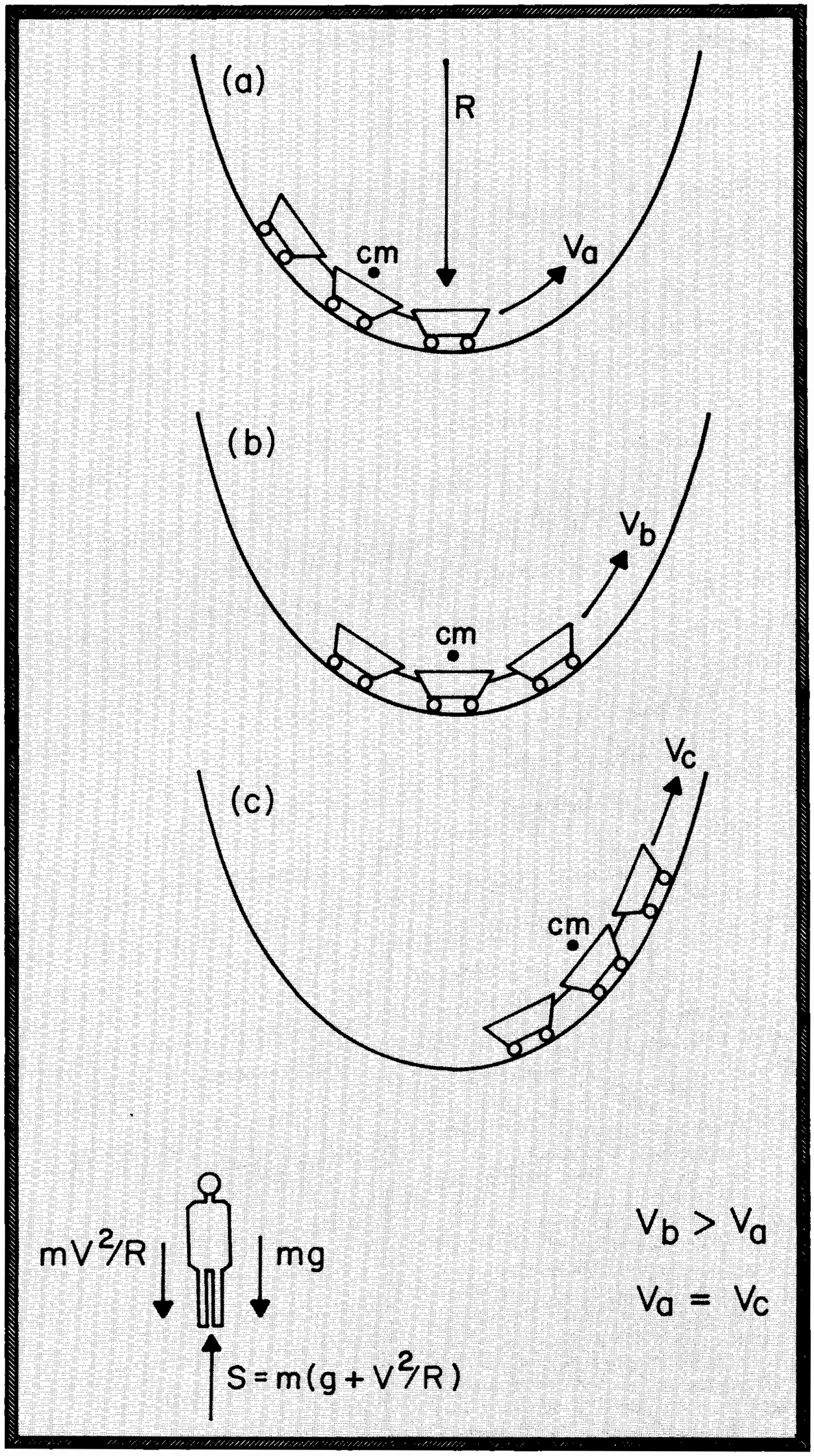
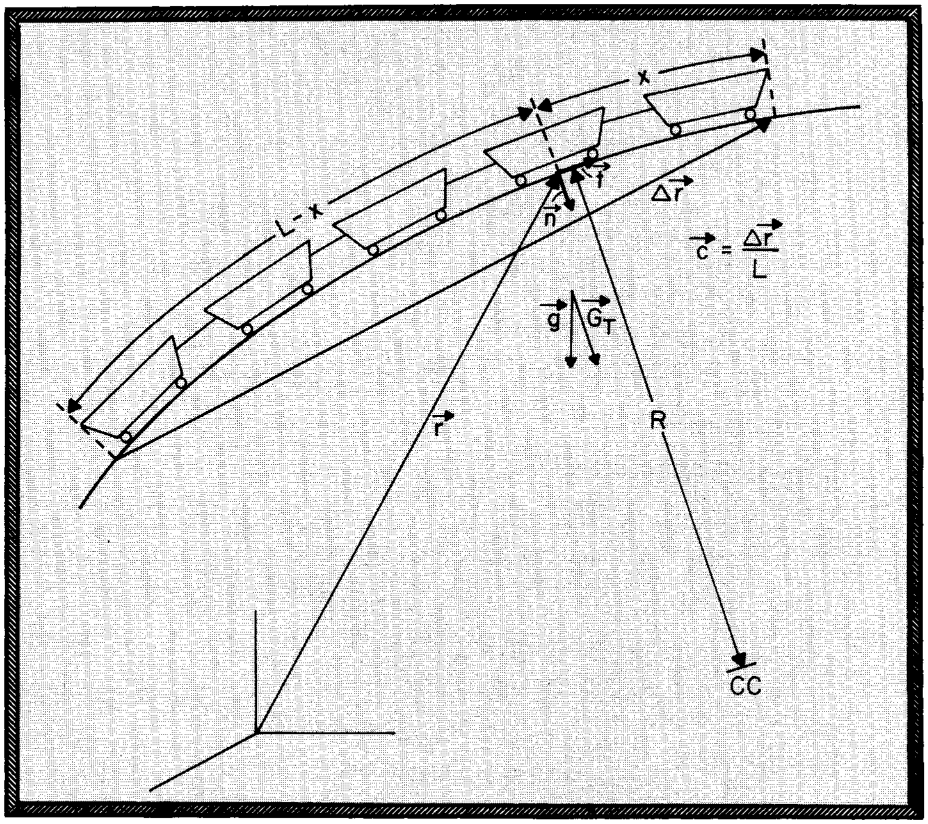



**Figure 3. Coaster train at the bottom of a symmetric hill: (a) first car at the bottom; (b) middle car at the bottom; (c) last car at the bottom. The radius of curvature is R. The position of the center of mass is cm. The cm is higher in cases (a) and (c) than (b), so, by conservation of energy, the speed V is greater in (b) than in (a) or (c). The vertical component of the seat force is $S = m(g + V^2/R)$. The inertial force $mV^2/R$ pushing the passenger down into the seat is greatest for the middle car.**

**Figure 4. Geometry for analysis of the seat force in a coaster train. The train has a total length L. The position vector of a point on the track is r. The tangent vector at r is t, and the normal vector at r is n. The radius of curvature at r is R, with the center of curvature being at CC. The transverse gravity at r is $G_T$. The chord vector from the rear to the front of the train is △r, and the dimensionless chord vector is c = △r/L. The speed of the train is V. The seat force at r is given by S = m[(V²/R)n − $G_T$ + g·(c−t)t].**




How large are these effects we've been discussing? We can get a simple answer using the approximate formulas developed in the Appendix. At a hilltop, the inertial force tending to lift us out of our seats is $mV^2/R$. We use equations (1) and (A-16) to calculate the difference in this force for the first (or last) car and the middle car. We find

$$m\left(\frac{V^2}{R}\right)_{\substack{\text{first}\\ \text{last}}} - m\left(\frac{V^2}{R}\right)_{\text{middle}} \approx \frac{mg}{4}\left(\frac{L}{R}\right)^2. \qquad (2)$$

The tighter the hill (smaller R) or the longer the train (larger L), the bigger the difference between the cars. If, for example, the train length equals the radius of curvature, then the seat force difference is 0.25 g's. If the ride has been designed so that passengers in the middle car are free-floating at the top of the hill, then passengers in the first and last cars feel a downward force from the lapbar or a harness equal to ¼ of their weight.

Although hill bottoms aren't as exciting as hilltops, they are still interesting. Figure 3 shows three consecutive positions of a coaster train at the bottom of a symmetric hill: (a) the first car at the bottom, (b) the middle car at the bottom, and (c) the last car at the bottom.

As we saw in Part I, the inertial force $mV^2/R$ at the hill bottom works with gravity and tends to push us down in our seats. Using equation (1) and looking at the position of the center of mass, we conclude that the speed is greatest for the middle car. That means we'll "pull" the most g's in the middle car. (This may well be the middle car's only claim to distinction!) The first and last cars will provide equal g's, but less than the middle car's. Quantifying the difference, we find the middle car provides more g's than the first and last cars by an amount $(1/4)(L/R)^2$. So, for a fairly tight curve with L = R, we get 0.25 g's more in the middle car.

Even though the first and last cars provide the same downward force at the hill bottom, they aren't equivalent in all respects, as we shall see next.

## Why Your Glasses Come Off

The extreme reductionist view of a coaster ride is that the entire experience is completely described by the time-varying force exerted on a passenger by the coaster car. A more moderate view is that our experience has its origin in this force and that we can understand the experience better if we know how this force is determined by the coaster geometry. We discussed this seat force **S** in Parts I and II of this primer. As we saw there, it is completely determined by the mass of the passenger m, the acceleration of gravity **g**, and the acceleration of the coaster car **a**:

$$\mathbf{S} = m(\mathbf{a} - \mathbf{g}) \qquad (3)$$

To make this formula useful, we need to express the acceleration **a** explicitly in terms of the train and track geometry and the coaster speed V. The calculations necessary to do this are given in the Appendix. The result is

$$\mathbf{S} = m[(V^2/R)\mathbf{n} - \mathbf{G_T} + \mathbf{g}\cdot(\mathrm{c} - \mathbf{t})\mathbf{t}] \qquad (4)$$

Here R is the radius of curvature of the track, **n** is a unit vector perpendicular to the track and pointing toward the center of curvature, **t** is a unit vector parallel to the track and pointing in the direction of motion, $\mathbf{G_T}$ is the component of gravity in the plane perpendicular to the track, and c is the vector pointing from the rear of the train to the front of the train, divided by L, the length of the train. These quantities are shown in Figure 4.

What's new here? How does this coaster-train seat force differ from the one we derived for a single coaster car [equation (7) of Part II]? The first two terms have the same form in both cases. However, for the single coaster car, there is no seat force component in the tangential direction, whereas equation (4) for the train contains a tangential force term, namely $\mathbf{g}\cdot(\mathrm{c} - \mathbf{t})\mathbf{t}$. Thus for a single car, the seat force comes from the bottom or the side, but for a train there can be an additional component, parallel to the track, from the back of the seat or from a restraining harness or lapbar. We already had a hint of this possibility in our earlier brief discussion of the coupling force between cars, because this force is parallel to the track. It's this force that accounts for the dynamical difference between the first and last cars, as we shall see next.

Consider first a special case. If the coaster track isn't curved, then c = t, and the geometric quantities **n**, t, and R are the same at all points of the train. In that case, the seat force **S** is the same at all points. Thus our formula verifies the intuitively obvious fact that car-to-car differences are induced by track curvature. When the track is curved, the quantities **n**, t, R, and $\mathbf{G}_T$ all vary along the track so that, at any given instant, passengers in different cars are experiencing different forces.

However, this snapshot view isn't the most interesting way to look at the situation. It's more revealing to explore what happens to passengers in different cars as they pass a given point on the track. We did this to some extent in our earlier discussion of variations of centrifugal force $m(V^2/R)\mathbf{n}$ from car to car. The second term in the seat force equation, the transverse gravity force $m\mathbf{G}_T$, is the same for all cars at a given point (it depends only on the mass m, gravity **g**, and the tangent vector t). The term in **S** that remains to be discussed is the tangential component of the seat force $S_t = m\mathbf{g}\cdot(c-t)$. As shown in the Appendix, this force component for the first, middle, and last cars, at a given point on the track with unit normal **n** and radius of curvature R, is

$$S_t = -(mL\mathbf{g}\cdot\mathbf{n})/(2R) \text{ (first car)}$$

$$S_t = 0 \text{ (middle car)} \qquad (5)$$

and

$$S_t = (mL\mathbf{g}\cdot\mathbf{n})/(2R) \text{ (last car)}$$

If $S_t$ is positive, the seat force is in the direction of the motion. Such a force is provided by the back of the seat, and the sensation for the passenger is that of being pushed back into the seat.

If, on the other hand, $S_t$ is negative, the seat force is backwards and therefore must be provided by a lapbar, a seat belt, or some contact with the front of the car. The sensation for the passenger is that of sliding forward in the car. We see from equation (5) that, once again, the middle car is the least interesting; it's tangential component of seat force is zero. The first and last cars have tangential seat forces of the same magnitude, but of opposite sign. To say more about the sign, we must specify the track geometry more completely.

Consider the situation at a hilltop, where **n** points straight down and is parallel to **g**. Then the first car has a tangential seat force $S_t = -(mgL)/(2R)$. Because this force is negative, we tend to slide forward in the first car as we top the hill. Anything that is not tied down also tends to slide forward—in particular, eyeglasses.

By contrast, when the last car tops the hill, the tangential seat force is positive and equal to (mgL)/(2R), and riders in that car are pushed back into their seats. The magnitude of this tangential force can be significant. If, for example, the hilltop has a 60-foot radius of curvature and the train is 30 feet long, then the magnitude of the tangential force in the first and last cars is 0.25 gs. At a hill bottom, the normal vector **n** is reversed, but everything else is the same, so the effect is reversed. As the front car reaches the hill bottom, we are pushed back into our seats. When the rear car reaches the bottom, we tend to slide forward, and our glasses attempt to come off. The poor middle car remains the dull one with no tangential force.

A summary of what we've learned about car-to-car differences is given in Table One (hilltops) and Table Two (hill bottoms).

| | **Effect of Normal Force** | **Effect of Tangential Force** | **Speed Change At Crest** |
|---|---|---|---|
| First car | Lifted up (most) | Slide forward | Slowing down |
| Middle car | Lifted up (least) | No effect | Neutral |
| Last car | Lifted up (most) | Pushed back | Speeding up |

Table 1. Effect of normal and tangential inertial forces on a passenger at the top of a symmetric hill.

| | **Effect of Normal Force** | **Effect of Tangential Force** | **Speed Change At Trough** |
|---|---|---|---|
| First car | Pushed down (least) | Pushed back | Speeding up |
| Middle car | Pushed down (most) | No effect | Neutral |
| Last car | Pushed down (least) | Slide forward | Slowing down |

Table 2. Effect of normal and tangential inertial forces on a passenger at the bottom of a symmetric hill.

## Closing Time

Those are sad words on a warm summer night when there's still one more coaster to ride. But there's always another day, and we have covered uphills, downhills, loops, and corkscrews in our three-part trip. It wasn't always easy going, but I hope this view through the window of dynamics has added a dimension to your coastering experiences. Although we heave dealt with some interesting questions, there are many more we've left unanswered. This is appropriate, because the highest purpose of a primer is not to answer questions, but to show how to ask questions. If this primer has caused you to ask new questions about your coaster experiences, I count it a success.

To all my fellow passengers out there: Thank you for joining me on these dynamical excursions!

## Acknowledgements

I thank Paul Ruben for providing photographs, for his editorial help, and for his friendly reminders of deadlines. Without his encouragement, this primer wouldn't have been written. I thank Nancy McGrath for her very helpful editorial suggestions on Part III. I thank Shari Harwell for transforming my rough sketches into the illustrations for all three parts. I thank my colleagues at the University of Rochester for their patience in hearing my various dynamical explanations of the coaster experience.

I dedicate this primer to Alice Andre-Clark—daughter, friend, and consummate coaster companion.

## Appendix—Engineer's Corner

Although you all are welcome to this corner with its cactus garden of equations, it is primarily for you kindred spirits who find happiness in a quantitative description of the world. Here we develop a mathematical model for the motion of coaster trains and for the seat force on the passenger. As in all mathematical models, we must pay for increased realism with increased complexity. Because our purpose here is insight rather than numerical accuracy, we keep the complexity to a minimum by choosing the simplest possible model. The geometric features which we are about to describe are shown in Figure 4. The track is defined by the position vector $\mathbf{r}$ as a function of arc length s along the track: $\mathbf{r} = \mathbf{r}(s)$. Then the unit tangent vector t, the unit normal $\mathbf{n}$ pointing to the center of curvature, and the radius of curvature are obtained from standard formulas in differential geometry:

$$\mathbf{t} = d\mathbf{r}/ds \qquad \mathbf{n}/R = d\mathbf{t}/ds \qquad \text{(A-1)}$$

We assume that the train and the passengers have total mass M distributed uniformly over a length L, and we ignore the transverse dimensions of the train compared to L. The train in our model, then, is like a flexible wire that follows the track. The next model up in the scale of complexity would treat the coaster cars as rigid bodies with rotational as well as translational kinetic energy. In our model, the position of the train can be specified completely by the arc length at the front of the train, $s = s_f(t)$, so the position of the front is $\mathbf{r}_f = \mathbf{r}(s_f)$, and the position of the rear is $\mathbf{r}_r = \mathbf{r}(s_r)$, where the arc length of the rear position is $s_r = s_f - L$. The speed of the train is

$$V = ds_f/dt = ds_r/dt \qquad \text{(A-2)}$$

As in Parts I and II, we ignore friction. Then we can use conservation of energy to relate speed and position. The kinetic energy of the train is $MV^2/2$, and the gravitational potential energy is $-M\mathbf{g}\cdot\mathbf{r}_c$, where $\mathbf{g}$ is gravity and $\mathbf{r}_c$ is the position of the center of mass, given by the integral

$$\mathbf{r}_c(t) = \frac{1}{L}\int_{s_r(t)}^{s_f(t)} \mathbf{r}(s)ds \cdot \qquad \text{(A-3)}$$

Conservation of energy is then expressed by

$$(MV^2/2) - M\mathbf{g}\cdot\mathbf{r}_c = E \qquad \text{(A-4)}$$

where E is the constant energy. We introduce a coordinate system so that z is positive upward. If $z = H$ is the top of the lift hill, where the velocity V is essentially zero, then $E = MgH$, and equation (A-4) becomes

$$V^2 = 2g(H - z_c) \qquad \text{(A-5)}$$

where $z_c$ is the z-coordinate of the center of mass $\mathbf{r}_c$. Thus at any point $\mathbf{r}$ we can calculate $z_c$ from equation (A-3) and the speed V from equation (A-5).

Now consider the seat force on a passenger of mass m. As shown in Part I, this seat force $\mathbf{S}$ is given by

$$\mathbf{S} = m(\mathbf{a} - \mathbf{g}) \qquad \text{(A-6)}$$

where $\mathbf{a}$ is the acceleration of the train at the location of the passenger. On a curved track, $\mathbf{a}$ is different at different points along the train, so not all passengers experience the same force at a given instant. To make the expression (A-6) useful, we must calculate the acceleration $\mathbf{a}$. The normal acceleration is given by $(V^2/R)\mathbf{n}$. To calculate the tangential acceleration $a_t = dV/dt$, we differentiate equation (A-4) with respect to time to get

$$V\frac{dV}{dt} = \mathbf{g}\cdot\frac{d\mathbf{r}_c}{dt} \quad . \qquad \text{(A-7)}$$

By differentiating the integral (A-3) we can get an expression for $d\mathbf{r}_c/dt$:

$$d\mathbf{r}_c/dt = \mathbf{c}V$$

where (A-8)

$$\mathbf{c} = (\mathbf{r}_f - \mathbf{r}_c)/L$$

is a dimensionless chord vector. Then the tangential acceleration is given by $a_t = \mathbf{g}\cdot\mathbf{c}$, and the vector acceleration is

$$\mathbf{a} = (\mathbf{g}\cdot\mathbf{c})\mathbf{t} + (V^2/R)\mathbf{n} \qquad \text{(A-9)}$$

As in Part II, we introduce the transverse gravity $\mathbf{G}_T$, which is the projection of $\mathbf{g}$ onto the plane transverse to the tangent vector $\mathbf{t}$:

$$\mathbf{G}_T = \mathbf{g} - (\mathbf{g}\cdot\mathbf{t})\mathbf{t} \qquad \text{(A-10)}$$

By combining equations (A-6), (A-9), and (A-10), we get the following form for the seat force:

$$\mathbf{S} = m[(V^2/R)\mathbf{n} - \mathbf{G}_T + \mathbf{g}\cdot(\mathbf{c} - \mathbf{t})\mathbf{t}] \qquad \text{(A-11)}$$

Compare this with the formula for a single coaster car [equation (7) of Part II]. In that case, only the terms $(V^2/R)\mathbf{n}$ and $\mathbf{G}_T$ are present. The main new feature here is the tangential component of the seat force, $\mathbf{g}\cdot(\mathbf{c}-\mathbf{t})\mathbf{t}$. Of course, as we have already discussed, the speed to be used in the formula (A-11) is different for a coaster train than for a single uncoupled car at the same track point.

In principle, the seat force can be calculated at any point from the collection of formulas we've just discussed. The procedure is intricate, however, and requires a modest computer program for a general track geometry. This complexity clouds somewhat our view of the new tangential force. Let's sacrifice a little accuracy for insight and develop a simple approximation for this force. We base the approximation on a Taylor series expansion about s. With the aid of equations (A-1), we get a three-term approximation:

$$\mathbf{r}(s + \triangle s) \approx \mathbf{r}(s) + \triangle s\,\mathbf{t} + \frac{1}{2}(\triangle s)^2\frac{\mathbf{n}}{R} \quad . \qquad \text{(A-12)}$$

We now suppose that the front of the train is located an arc length distance x beyond $\mathbf{r}$, as shown in Figure 4. Then, by using equation (A-12) to calculate the normalized chord c, we get for the tangential seat force

$$S_t = m\mathbf{g}\cdot(\mathbf{c} - \mathbf{t}) \approx m\frac{\mathbf{g}\cdot\mathbf{n}}{R}\left(x - \frac{L}{2}\right) \quad . \qquad \text{(A-13)}$$

By varying x in this equation (front, $x = 0$; middle, $x = L/2$; rear, $x = L$), we can determine how the tangential force differs for passengers in different cars as they pass a given point on the track. We get

$$S_t = -\frac{m\mathbf{g}\cdot\mathbf{n}\,L}{2R} \qquad \text{(first car)}$$

$$S_t = 0 \qquad \text{(middle car)}$$

and

$$S_t = \frac{m\mathbf{g}\cdot\mathbf{n}\,L}{2R} \qquad \text{(last car)} \qquad \text{(A-14)}$$

We discussed the significance of equations (A-14) in the main text for hilltops and hill bottoms. Let's make a more general observation here. For a center of curvature below the track, $\mathbf{g}\cdot\mathbf{n}$ is positive, so we slide forward in the first car and are pushed back into our seat in the last car. For a center of curvature above the track, $\mathbf{g}\cdot\mathbf{n}$ is negative, and the effects are reversed: we slide forward in the last car and are pushed back in the first car.

We may also use the Taylor series (A-12) in the integral (A-3) to get a simple approximation for the position of the center of mass. We again suppose that the front of the train is an arc-length distance x beyond the point $\mathbf{r}$. Then we get

$$\mathbf{r}_c = \mathbf{r} + (x - L/2)\mathbf{t} + \frac{(3x^2 - 3xL + L^2)}{6}\frac{\mathbf{n}}{R} \qquad \text{(A-15)}$$

Again we may apply this to the first car ($x = 0$), middle car ($x = L/2$), and the last car ($x = L$). We get

$$\mathbf{r}_c = \mathbf{r} - \frac{L}{2}\mathbf{t} + \frac{L^2}{6}\frac{\mathbf{n}}{R} \qquad \text{(first car)} \qquad \text{(A-16)}$$

$$\mathbf{r}_c = \mathbf{r} + \frac{L^2}{24}\frac{\mathbf{n}}{R} \qquad \text{(middle car)}$$

$$\mathbf{r}_c = \mathbf{r} + \frac{L}{2}\mathbf{t} + \frac{L^2}{6}\frac{\mathbf{n}}{R} \qquad \text{(last car)}$$

The formulas (A-14) and (A-16), which are based on the approximation (A-12), should be reasonably accurate as long as the track geometry does not change too rapidly with position. For improved accuracy, we could keep more terms in the Taylor expansion (A-12). For rapidly changing track geometry, we would have to use the more exact formulas (A-3) for $\mathbf{r}_c$ and (A-11) for $\mathbf{S}$, and in general, this would require some numerical work on a computer. ☆